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On the Stefan Problem and Some of its Generalizations

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Introduction

At the end of the nineteenth century the physician Joseph Stefan proposed a model for the melting of polar ices and analyzed several problems concerning fusion and solidification processes. During the last century, especially in the second half, this model received a good deal of attention and several modifications; however, due to the importance of the work of Stefan, it has been named after him, and nowadays it is known as the *Stefan problem*.

Essentially this problem consists in solving the heat equation in the two subdomains containing respectively the solid and the liquid phases, taking into account of an *interface condition*: this condition states that the normal velocity of the interface separating the solid and the liquid phases, is proportional to the jump of the gradient of the temperature. This interface condition is usually called *Stefan condition*.

Of course the temperature field θ is an unknown, but this is not the only one: also the interface S which separates the two phases must be determined. Hence the Stefan problem is a typical example of *free boundary problem*.

The formulation of the Stefan problem in the form that we have outlined above is usually called *strong formulation*. In general this formulation is not well-posed. In this dissertation we will deal with another formulation of the Stefan problem, usually termed *weak formulation of the Stefan problem*.

This alternative formulation is constructed on the basis of the consideration that the aforementioned interface S is not a material surface and its evolution does not represent any motion of particles. Instead, in dealing with phase transition processes at a macroscopic length scale, it is natural to assume that the two phases are not separated by a sharp interface, but there exists a third region where a fine solid-liquid mixture appears, the so-called *mushy region*. Therefore we can define a second variable χ, the *phase function*, by setting $\chi \equiv 1$ in the liquid phase, $\chi \equiv -1$ in the solid phase, and we allow χ to attain any values in $[-1, 1]$ in the mushy region. In this new setting, denoting by f a space dependent heat source, the weak formulation of the Stefan problem essentially reads

$$\frac{\partial(\theta+\chi)}{\partial t} - \Delta\theta = f, \qquad \text{in } Q := \Omega\times]0,T[, \tag{0.1}$$

$$\chi \in \operatorname{sign}(\theta) \qquad \text{in } Q. \tag{0.2}$$

Here Ω is the domain containing the two phase-material, $[0,T]$ is the time interval where the evolution is studied, and θ and χ are the unknowns of the problem. The first equation is the energy balance, whereas the second condition means that at every point (x,t) of Q, the inclusion $\chi(x,t) \in \operatorname{sign}(\theta(x,t))$ must hold, where sign denotes the multivalued mapping defined by $\operatorname{sign}(r) := 1$ if $r > 1$, $\operatorname{sign}(r) := -1$ if $r < 1$, $\operatorname{sign}(0) = [-1,1]$. This condition is natural and is in agreement with the concept of mushy region. Let us clarify that, despite the terminology, the two formulations represent two different problems. Henceforth we will refer to the weak formulation simply as to the *Stefan problem*, and also condition (0.2) is called *Stefan (equilibrium) condition*.

The weak formulation of the Stefan problem is due to Kamenomostskaya and Oleinik who stated it around 1960 (cf. [20] and [29]). Existence and uniqueness results for problem (0.1)–(0.2) have been proved by several authors. The first significant result obtained in a variational setting is due to Damlamian who in 1977 solved the problem by means of techniques from convex analysis (see [14]).

In this dissertation we deal with some generalizations of the Stefan problem which allow to consider physical phenomena neglected by the model (0.1)–(0.2).

The first generalization is based on the fact that the Stefan condition (0.2) is an equilibrium condition, whereas we would like to consider situations characterized by a *non equilibrium condition*, that can arise in presence of superheating or undercooling phenomena. To this aim in 1985 Visintin proposed to replace (0.2) by the following *relaxation dynamics*

$$\varepsilon\frac{\partial\chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q, \tag{0.3}$$

where ε is a small positive relaxation parameter. Equation (0.1), coupled with (0.3) is called *Stefan problem with phase relaxation*, and in paper [36] it is proved that this problem, endowed with suitable initial and boundary conditions, is well posed in the framework of Sobolev spaces, and that its solution converges in a suitable fuctional space to the solution of the Stefan problem when ε approaches zero.

In this dissertation we study an alternative model of phase relaxation, i.e. we replace (0.2) by the inclusion

$$\varepsilon\frac{\partial\chi}{\partial t} + \chi \ni \operatorname{sign}(\theta) \qquad \text{in } Q, \tag{0.4}$$

and we couple it with the energy balance (0.1). The model obtained in this way, although similar, is non equivalent to the classical phase relaxation (0.3). Following a paper of the author (cf. [30]) we provide a physical justification of this model that is based on a probabilistic interpretation of the phase transition which can be found in [38]. Moreover we couple (0.1), (0.4) with suitable initial and boundary conditions, we prove an existence and uniqueness result for the resulting problem in a Sobolev space setting, and we show that in a suitable topology the solution of (0.1)–(0.4) converges to the solution of the Stefan problem as ε tends to zero. We point out that this asymptotic analysis is relevant because in most common physical situations, the relaxation parameters are very small, so that it is reasonable to assume that the Stefan model is a good approximation of the relaxed models.

We consider also another generalization concerning the energy balance (0.1). Let us observe that, if $\mathbf{q}$ denotes the heat flux, then (0.1) is obtained from the standard relation

$$\frac{\partial(\theta+\chi)}{\partial t} + \operatorname{div}\mathbf{q} = f \qquad \text{in } Q, \tag{0.5}$$

where we assume the Fourier constitutive heat flux law

$$\mathbf{q} = -\nabla\theta \qquad \text{in } Q. \tag{0.6}$$

The Fourier heat conduction law (0.6) has the well known feature that it predicts that thermal disturbances propagate at infinite speed. Although in a lot of cases this is not a bad inconvenient, there are some materials (cf., e.g., [10]) where the Fourier law does not seem to be very satisfactory. The first tentative to overcome this problem is due to Cattaneo who proposed in 1948 (see [7]) to substitute the Fourier law with the relaxation for the heat flux given by

$$\alpha\frac{\partial\mathbf{q}}{\partial t} + \mathbf{q} = -\nabla\theta \qquad \text{in } Q. \tag{0.7}$$

Following a paper by P. Colli and the author (see [13]), we couple (0.5), (0.7) with (0.3) and obtain the so called *phase relaxation problem with Cattaneo-Maxwell heat flux law*. We complement this problem with rather general initial and boundary conditions and we prove an existence result. Finally we show that if $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ is an arbitrary solution of (0.5), (0.7), (0.3) for any $\alpha, \varepsilon > 0$, then the sequence $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon})$ converges to the solution of the Stefan problem (0.1)–(0.2) in a natural functional space. Let us remark that we are able to prove that such convergence holds without assuming any relation between the two parameters α and ε. Moreover the proof is not trivial essentially because we are not able to prove any uniqueness result for the relaxed problem, so that it is not straightforward to recover a priori estimates that are valid for *any* solution of problem (0.5), (0.7) , (0.3).

Now let us briefly outline the content of the dissertation. Chapter I is devoted to the weak formulation of the Stefan problem. After introducing the physical problem, we present all the tools necessary to prove an existence and uniqueness theorem for (0.1)–(0.2). We solve it by means of an *approximation-a priori estimate-limit* procedure, taking advantage of classical results about weak solution of the heat equation. For this procedure we essentially follow a paper by Colli and Grasselli (cf. [12]). Since the Stefan problem involves the maximal monotone sign graph, we also devote a section to the main properties of maximal monotone graphs in $\mathbb{R}^2$. This allows us to solve the Stefan problem without employing the big machinery of the theory of maximal monotone operators in Hilbert space, so that we can detail all the proofs.

In the second chapter we first present the classical phase relaxation introduced in [36] and state the relative existence and convergence theorems. Also in this case we detail all the proofs exploiting the preparatory material shown in Chapter I. Then we illustrate the more recent model (0.1), (0.4) and prove the results we have mentioned before. In order to prove existence of solution of the model we employ a fixed-point technique for multivalued mappings.

In Chapter III we recall the heat conduction model by Cattaneo and Maxwell and we couple it with the relaxed Stefan problem. We prove the relative existence theorem by means of an elliptic regularization and then we recover some a priori estimate which hold for any solution of the relaxed system. These bounds allow us to recover the solution of the Stefan problem as the limit of the solution of the relaxed system as the two relaxation parameters go to zero.

Finally we collect in an Appendix some technical tools from analysis (some of them very deep and important) which we have exploited during the previous chapters. The aim of this appendix is essentially to fix some notations and there is not any claim of completeness.

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List of notations

$\mathscr{P}(X)$, the collection of all subsets of the set X

Id_X, identity map in the set X

$\mathbb{N}$, the set of positive integer (including zero), i.e. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{N}_*$, the set of strictly positive integer (not including zero), i.e. $\mathbb{N}_* = \{1, 2, 3, \ldots\}$

$\widetilde{\mathbb{R}}$, the extended real line, i.e. $\widetilde{\mathbb{R}} = [-\infty, \infty]$

$f(t_0+)$, the right-limit of f as $t \to t_0+$; analogous notation for left-limits

$\mathrm{Lip}(f)$, the Lipschitz constant of a function f

$\mathrm{supp} f$, the support of a function f

$(\cdot, \cdot)_H$, inner product of a Hilbert space H

E', topological dual of a topological vector space E

$\mathscr{L}(E, F)$, the space of linear bounded operators from E into F

$u_n \rightharpoonup u$, the sequence u_n is weakly converging to the element u

$u_n \overset{*}{\rightharpoonup} u$, the sequence u_n is weakly-star converging to the element u

$\mathcal{L}^d$, d-dimensional Lebesgue measure

$\mathcal{H}^d$, d-dimensional Hausdorff measure

Contents

Chapter I

The Stefan problem

In this chapter we introduce the weak formulation of the Stefan model for phase transition phenomena in solid-liquid systems. Then we review the existence and uniqueness theorems for weak solutions of the heat equation, and we recall some theorems about evolution equations governed by a special class of maximal monotone operators. All these results will be exploited to solve the Stefan problem in its weak formulation. All the methods and results presented are classical, even if some theorems are stated in a slightly more general form than those already existing in the literature. Since the Stefan problem involves only a particular kind of monotone operators, we recall and use only the notion of monotone graph in $\mathbb{R}^2$. In this framework we are able to detail all the proofs without employing the big machinery of the theory of monotone operators in Hilbert space.

I.1 The Stefan model

Within this section we introduce the strong and the weak formulations of Stefan problem. Let us consider a physical system composed of a material exhibiting two phases, for example an ice-water system. We assume that the material is homogeneous and isotropic and that it is contained in bounded domain $\Omega \subseteq \mathbb{R}^3$. We are interested in the study of the thermal evolution of this material during a time interval $[0, T]$, where T is a positive finite final time. Let us denote by $\theta(x, t)$ the relative temperature of the material in the point $x \in \Omega$ at the time $t \in]0, T[$; let us also assume that a heat source is present in Ω at any time of the interval $]0, T[$:

- $g(x, t)$ heat supply in the point $x \in \Omega$ at the time $t \in]0, T[$.

Then it is well known that the heat propagation in the material is described by the relation

$$C_v \frac{\partial \theta}{\partial t} + \operatorname{div} \mathbf{q} = g \qquad \text{in } Q := \Omega \times]0, T[, \tag{1.1}$$

where we define:

- C_v heat capacity per unit volume;
- $\mathbf{q}$ heat flux.

We need a constitutive law relating the heat flux $\mathbf{q}$ and the temperature θ. The most common assumption is given by the classical *Fourier heat conduction law*:

$$\mathbf{q} = -k \nabla \theta \qquad \text{in } Q \tag{1.2}$$

where we have:

- k thermal conductivity.

Let us observe that the positive coefficients C_v and k may depende on (x, t) and in general also on the temperature θ. Coupling (1.1) and (1.2) we get the so called *heat equation*

$$C_v \frac{\partial \theta}{\partial t} - \operatorname{div}(k \nabla \theta) = g \qquad \text{in } Q. \tag{1.3}$$

In order to describe physical processes where a phase change occurs, now we introduce, starting from the heat equation, a model of phase transition due to Josef Stefan, who formulated this model at the end of nineteenth century during his studies on the melting of polar ices (see [35]). We assume that at any time t the domain Ω can be divided into two subsets containing respectively the liquid and the solid phase. We define

- $\Omega_\ell(t)$, $\Omega_s(t)$ open subsets of Ω occupied by the solid and liquid phases, respectively, at the time t;
- $Q_\ell := \bigcup_{t \in]0,T[} (\Omega_\ell(t) \times \{t\})$, $\qquad Q_s := \bigcup_{t \in]0,T[} (\Omega_s(t) \times \{t\})$;
- S_t *interface* separating $\Omega_\ell(t)$ and $\Omega_s(t)$;
- $S := \bigcup_{t \in]0,T[} (S_t \times \{t\})$;
- $\boldsymbol{\nu} \in \mathbb{R}^4$ unit vector normal to S, pointing outwards Ω_ℓ (i.e. from liquid to solid); we also set $\boldsymbol{\nu} =: (\boldsymbol{\nu}_x, \nu_t)$ with $\boldsymbol{\nu}_x \in \mathbb{R}^3$ and $\nu_t \in \mathbb{R}$.

We will assume that S is a smooth manifold in $\mathbb{R}^4$ and we suppose that $\theta = 0$ is the critical temperature of phase transition, i.e.

$$\begin{cases} \theta(x,t) > 0 & \forall (x,t) \in Q_\ell \\ \theta(x,t) = 0 & \forall (x,t) \in S \\ \theta(x,t) < 0 & \forall (x,t) \in Q_s \end{cases} . \tag{1.4}$$

The *Stefan model* then reads as follows

$$C_{v,\ell}\frac{\partial\theta}{\partial t} - \operatorname{div}(k_\ell \nabla\theta) = g \qquad \text{in } Q_\ell, \tag{1.5}$$

$$C_{v,s}\frac{\partial\theta}{\partial t} - \operatorname{div}(k_s \nabla\theta) = g \qquad \text{in } Q_s, \tag{1.6}$$

$$k_\ell \nabla\theta_\ell \cdot \boldsymbol{\nu}_x - k_s \nabla\theta_s \cdot \boldsymbol{\nu}_x = L\nu_t \qquad \text{on } S. \tag{1.7}$$

Here θ_ℓ, θ_s, $C_{v,\ell}$, $C_{v,s}$, k_ℓ, and k_s have the obvious meanings and we define

- L latent heat of the material.

The first two equations (1.5) and (1.6) are the heat equations in the liquid and solid part, respectively. The condition (1.7) is called *Stefan condition.* Essentially (1.7) states that the normal velocity of the interface S at the time t is proportional to the jump of the gradient of the temperature. The *Stefan problem* consists in finding a function θ and a manifold S satisfying (1.5)–(1.7), coupled with suitable initial and boundary conditions. This kind of problems are called *free boundary problems*, because also the interface S is unknown. Conditions (1.5)–(1.7) are also termed *strong formulation of the Stefan problem* and a pair (θ, S) is said to be a *strong solution* of the Stefan problem if θ and S are sufficiently smooth and (1.5)–(1.7) hold.

We do not give here more precise definitions because we are interested in a weaker form of the Stefan problem. Incidentally we notice that in general the problem (1.5)–(1.7) is not well-posed problem (cf. [37], p. 93), and there are still many question left open, for example such as the regularity of the free boundary S. For a survey on this subject we refer to [26].

Now let us deduce this weaker formulation of the Stefan problem. We define the *phase function* $\chi : Q \longrightarrow \mathbb{R}$ by setting

$$\chi(x,t) := \begin{cases} 1 & \text{if } (x,t) \in Q_\ell \\ -1 & \text{if } (x,t) \in Q_s \end{cases} . \tag{1.8}$$

Hence (1.5)–(1.6) can be rephrased by writing

$$C_{v,\ell}\frac{\partial\theta}{\partial t} + \frac{L}{2}\frac{\partial\chi}{\partial t} - \operatorname{div}(k_\ell \nabla\theta) = g \qquad \text{in } Q_\ell,$$

$$C_{v,s}\frac{\partial\theta}{\partial t} + \frac{L}{2}\frac{\partial\chi}{\partial t} - \operatorname{div}(k_s \nabla\theta) = g \qquad \text{in } Q_s.$$

Thus we are led to consider a relation like

$$C_v \frac{\partial \theta}{\partial t} + \frac{L}{2}\frac{\partial \chi}{\partial t} - \operatorname{div}(k\nabla\theta) = g \qquad \text{in } Q. \tag{1.9}$$

We want to show, at least in formal way, that the previous equation holds in the sense of distributions (cf. Section A.1 of the Appendix). Therefore let us multiply the equation (1.9) by a function $\varphi \in C_c^\infty(Q)$ and integrate over Q. Then we get

$$\int_Q C_v \frac{\partial \theta}{\partial t}\varphi + \int_Q \frac{L}{2}\frac{\partial(\chi\varphi)}{\partial t} - \int_Q \frac{L}{2}\chi\frac{\partial \varphi}{\partial t} - \int_Q \operatorname{div}(k\nabla\theta)\varphi = \int_Q g\varphi \tag{1.10}$$

(in writing the integrals we have omitted the symbol $dxdt = d\mathcal{L}^4$, the Lebesgue measure in $\mathbb{R}^4$). Now, using the Green formula for the space variables, observing that φ is zero on ∂Q, and using the Stefan condition (1.7), we infer that

$$\begin{aligned}\int_Q \operatorname{div}(k\nabla\theta)\varphi d\mathcal{L}^4 &= \int_{Q_\ell} \operatorname{div}(k_\ell\nabla\theta)\varphi d\mathcal{L}^4 + \int_{Q_s} \operatorname{div}(k_s\nabla\theta)\varphi d\mathcal{L}^4\\ &= -\int_{Q_\ell} k_\ell\nabla\theta\cdot\nabla\varphi d\mathcal{L}^4 - \int_{Q_s} k_s\nabla\theta\cdot\nabla\varphi d\mathcal{L}^4\\ &\quad + \int_S (k_\ell\nabla\theta_\ell\cdot\boldsymbol{\nu}_x - k_s\nabla\theta_s\cdot\boldsymbol{\nu}_x)\varphi d\mathcal{H}^3\\ &= -\int_Q k\nabla\theta\cdot\nabla\varphi d\mathcal{L}^4 + \int_S L\nu_t\varphi d\mathcal{H}^3. \end{aligned}\tag{1.11}$$

Here $\mathcal{H}^3$ denotes the 3-dimensional Hausdorff measure on S (in other terms the surface measure on S). On the other hand, by an integration by parts in time, and due to (1.8), we find that

$$\begin{aligned}\int_Q \frac{L}{2}\frac{\partial(\chi\varphi)}{\partial t}d\mathcal{L}^4 &= \int_{Q_\ell} \frac{L}{2}\frac{\partial(\chi\varphi)}{\partial t}d\mathcal{L}^4 + \int_{Q_s} \frac{L}{2}\frac{\partial(\chi\varphi)}{\partial t}d\mathcal{L}^4 =\\ &= -\int_{Q_\ell} \frac{L}{2}\chi\varphi d\mathcal{L}^4 - \int_{Q_s} \frac{L}{2}\chi\varphi d\mathcal{L}^4\\ &\quad + \int_S \frac{L}{2}(1)\varphi\nu_t d\mathcal{H}^3 - \int_S \frac{L}{2}(-1)\varphi\nu_t d\mathcal{H}^3\\ &= \int_S L\varphi\nu_t d\mathcal{H}^3. \end{aligned}\tag{1.12}$$

Hence we finally find

$$\int_Q C_v \frac{\partial \theta}{\partial t}\varphi d\mathcal{L}^4 + \int_Q \frac{L}{2}\chi\frac{\partial \varphi}{\partial t}d\mathcal{L}^4 - \int_Q k\nabla\theta\cdot\nabla\varphi d\mathcal{L}^4 = \int_Q g\varphi d\mathcal{L}^4.$$

Thus

$$C_v \frac{\partial \theta}{\partial t} + \frac{L}{2} \frac{\partial \chi}{\partial t} - \operatorname{div}(k \nabla \theta) = g \qquad \text{in } \mathscr{D}'(Q). \tag{1.13}$$

In the above we did not define the phase function χ on the interface S, and in fact this is immaterial because it is understood that S is interpreted as sharp interface, and not as a material surface, i.e. its evolution does not represent motion of particles. However, if we want to work with a *macroscopic* lenght-scale, it is more reasonable to construct a model which takes account of a very fine solid-liquid mixture, which typically appears at macroscopic scales. Thus we have to deal with the liquid region Q_ℓ, the solid region Q_s, and a third region where the solid-liquid mixture occurs. This third region is called *mushy region.* In order to model this situation, then it is natural to allow χ to attain the values between -1 and 1 whenever $\theta = 0$. Then we define the multivalued function $\operatorname{sign} : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ by setting

$$\operatorname{sign}(r) := \begin{cases} -1 & \text{if } r < 0 \\ [-1, 1] & \text{if } r = 0 \\ 1 & \text{if } r > 0 \end{cases}$$

and we replace the relation (1.8) with the following more general constitutive law

$$\chi \in \operatorname{sign}(\theta) \qquad \text{in } Q. \tag{1.14}$$

Of course the previous relation has to be interpreted pointwise, i.e. $\chi(x,t) \in \operatorname{sign}(\theta(x,t))$ for any $(x,t) \in Q$. Thus if we couple the equation (1.13) with the previous inclusion, we get the so called *weak formulation* of the Stefan problem:

$$C_v \frac{\partial \theta}{\partial t} + \frac{L}{2} \frac{\partial \chi}{\partial t} - \operatorname{div}(k \nabla \theta) = g \qquad \text{in } \mathscr{D}'(Q), \tag{1.15}$$

$$\chi \in \operatorname{sign}(\theta) \qquad \text{in } Q. \tag{1.16}$$

Throughout the sequel we will deal with a formulation stronger than (1.15)–(1.16), more precisely we will set our Stefan problem in a Sobolev space framework (see section A.1 for the main definitions), and we will couple the resulting equations with suitable initial and boundary conditions. Moreover we will make the simplifying, but common, assumption that C_v, k, and L are positive constant, and for simplicity we normalize them to 1. The problem we will solve can be formally written as

$$\frac{\partial(\theta+\chi)}{\partial t} - \Delta\theta = g \qquad \text{in } Q, \tag{1.17}$$
$$\chi \in \operatorname{sign}(\theta) \qquad \text{in } Q, \tag{1.18}$$
$$\theta = \theta_D \quad \text{on } \Gamma_D \times]0,T[, \qquad \frac{\partial\theta}{\partial \mathbf{n}} = \theta_N \quad \text{on } \Gamma_N \times]0,T[, \tag{1.19}$$
$$(\theta+\chi)(\cdot,0) = \theta_0 + \chi_0 \qquad \text{in } \Omega. \tag{1.20}$$

Here $\mathbf{n}$ is the outward unit vector, normal to $\partial\Omega$, $\{\Gamma_D, \Gamma_N\}$ is a partition of $\Gamma := \partial\Omega$, θ_D and θ_N are given functions defined respectively on $\Gamma_D \times]0,T[$ and $\Gamma_N \times]0,T[$, and $\theta_0, \chi_0 : \Omega \longrightarrow \mathbb{R}$ are the initial data for θ and χ. If we make the further assumption that θ_D is a sufficiently smooth function defined on the whole Q, and if we set $\overline{\theta}_0 := \theta_0$ and $f := g - \theta'_D - \Delta\theta_D$, then we can rewrite system (1.17)–(1.20) in terms of the new unknown $\overline{\theta} = \theta - \theta_D$, obtaining the new system

$$\frac{\partial(\overline{\theta}+\chi)}{\partial t} - \Delta\overline{\theta} = f \qquad \text{in } Q, \tag{1.21}$$
$$\chi \in \operatorname{sign}(\overline{\theta}+\theta_D) \qquad \text{in } Q, \tag{1.22}$$
$$\overline{\theta} = 0 \quad \text{on } \Gamma_D \times]0,T[, \qquad \frac{\partial\theta}{\partial \mathbf{n}} = \theta_N \quad \text{on } \Gamma_N \times]0,T[, \tag{1.23}$$
$$(\overline{\theta}+\chi)(\cdot,0) = \overline{\theta}_0 + \chi_0 \qquad \text{in } \Omega, \tag{1.24}$$

which presents the advantage to deal with homogeneous Dirichlet boundary condition, i.e. $\overline{\theta}$ is required to be zero on $\Gamma_D \times]0,T[$.

I.2 The heat equation

In this section we are going to study a weak formulation of the heat equation (1.3), where the physical constant coefficients are normalized to 1:

$$\frac{\partial\theta}{\partial t} - \Delta\theta = f \qquad \text{in } Q.$$

In order to solve this equation we will employ the so called *variational methods*. We follow essentially [15, Vol. Chapter XVIII], but see also [24]). Under suitable regularity assumptions on the data, different and more direct approaches can be used (see, e.g., [18] and [16]). Before giving the weak formulation of the problem, we display precisely all the assumptions on the data. We have

$$d \in \mathbb{N}_*, \quad \Omega \text{ is a bounded open and connected subset of } \mathbb{R}^d, \tag{2.1}$$
$$\Gamma := \partial\Omega \text{ is of Lipschitz class,} \tag{2.2}$$
$$\mathbf{n} \text{ is the outward normal unit vector to } \Omega, \tag{2.3}$$
$$\Gamma_D \text{ and } \Gamma_N \text{ are open subsets of } \Gamma \tag{2.4}$$
$$\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma, \quad \Gamma_D \cap \Gamma_N = \varnothing, \quad \overline{\Gamma}_D \cap \overline{\Gamma}_N \text{ is of Lipschitz class} \tag{2.5}$$
$$Q := \Omega \times]0, T[, \quad \text{where } T \in]0, \infty[. \tag{2.6}$$

Then we set

$$H := L^2(\Omega), \qquad V := H^1_{\Gamma_D}(\Omega), \tag{2.7}$$

where we recall that

$$H^1_{\Gamma_D}(\Omega) := \left\{ u \in H^1(\Omega) \ : \ u|_{\Gamma_D} = 0 \right\}$$

(cf. Section A.1.12 for the main properties of Sobolev spaces and traces). The spaces H and V are endowed with their usual inner product. Identifying H with its dual H' we obtain a Hilbert triplet

$$V \subseteq H \subseteq V'$$

with compact embeddings. The inner product in H will be denoted by $(\cdot, \cdot)$, whereas we use the usual brackets $\langle \cdot, \cdot \rangle$ for the duality pairing between V' and V. Moreover we set

$$\mathbf{H} := L^2(\Omega; \mathbb{R}^d). \tag{2.8}$$

We will further assume that

$$f \in L^2(0, T; V') + L^1(0, T; H), \tag{2.9}$$
$$u_0 \in H. \tag{2.10}$$

Finally let us define the operator $A \in \mathscr{L}(V, V')$ by setting

$$\langle A v_1, v_2 \rangle := \int_\Omega \nabla v_1 \cdot \nabla v_2, \qquad v_1, v_2 \in V. \tag{2.11}$$

If E is a Banach space, for a function $\phi :]0, T[\longrightarrow E$ we will use the notation ϕ' to denote the distributional derivative of u. Now we can state weak formulation of the heat equation.

Problem (H). Assume (2.1)–(2.10). Find a function u such that

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{2.12}$$
$$u' \in L^2(0,T;V') + L^1(0,T;H), \tag{2.13}$$
$$u' + Au = f \qquad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{2.14}$$
$$u(0) = u_0. \tag{2.15}$$

♠

Remark I.2.1. Notice that the initial condition (2.15) makes sense because of (2.12)–(2.13). Indeed, thanks to Proposition A.2.2, u is almost everywhere equal to a continuous function $\overline{u}$, so that the trace of u is well defined and is given by $\overline{u}(0)$.

We will solve Problem **(H)** by approximating it with a sequence of problems which are essentially a sequence of ordinary differential equations in some finite dimensional space. Then we will recover a solution of (2.12)–(2.15) as the limit of the sequence of the "approximated" solutions. For convenience we recall that equation (2.14) means that

$$\langle u'(s), v\rangle + \langle Au(s), v\rangle = \langle f(s), v\rangle \qquad \forall v \in V, \quad \text{for a.a. } s \in]0,T[. \tag{2.16}$$

Since V is separable, it admits a Hilbert basis $(e_n)_{n\in\mathbb{N}_*}$. Thus, if for any $n \in \mathbb{N}_*$ we set $V_n := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$, we have $V_n \subseteq V_{n+1}$ for all n and $V_\infty := \bigcup_n V_n$ is dense in V. Therefore, by the density of V in H, there exists a sequence $(u_{0,n})$ such that

$$u_{0,n} \in V_n \qquad \forall n, \qquad\qquad u_{0,n} \to u_0 \qquad \text{in } H. \tag{2.17}$$

Now we can present the approximation of Problem **(H)**.

Problem (H$_n$). Let $n \in \mathbb{N}_*$ and assume (2.1)–(2.10). Find a function u_n such that

$$u_n \in W^{1,1}(0,T;V_n), \tag{2.18}$$
$$\langle u_n'(s), v\rangle + \langle Au_n(s), v\rangle = \langle f(s), v\rangle \qquad \forall v \in V_n, \quad \text{for a.a. } s \in]0,T[, \tag{2.19}$$
$$u_n(0) = u_{0,n}. \tag{2.20}$$

♠

Proposition I.2.1. *Let $n \in \mathbb{N}_*$. Problem* $(\mathbf{H}_n)$ *has a unique solution.*

Proof. Problem $(\mathbf{H}_n)$ is clearly equivalent to find $u_n \in W^{1,1}(0,T;V_n)$ such that the following conditions hold

$$(u_n'(s), e_j) + \langle Au_n(s), e_j\rangle = \langle f(s), e_j\rangle \qquad \text{for a.a. } s \in]0,T[, \quad \forall j = 1,\dots,n, \tag{2.21}$$

$$u_n(0) = u_{0,n} \tag{2.22}$$

(recall that $\langle u_n'(s), e_j\rangle = (u_n'(s), e_j)$ because $u_n'(s) \in H$). In other terms we look for a map $y_n = (y_{n,1},\dots,y_{n,n}) \in W^{1,1}(0,T;\mathbb{R}^n)$ such that $u_n(s) = \sum_{i=1}^n y_{n,i}(s)e_i$ for $s \in]0,T[$ and (2.21)–(2.22) are satisfied. Let us set

$$a_{ij} := \langle Ae_j, e_i\rangle, \quad b_{ij} := (e_j, e_i), \qquad i,j = 1,\dots,n,$$

and define the $(n \times n)$-matrices A_n and B_n, the function $f_n :]0,T[\longrightarrow \mathbb{R}^n$, and the vector $y_{0,n} \in \mathbb{R}^n$ by

$$A_n := (a_{ij})_{ij}, \quad B_n := (b_{ij})_{ij},$$
$$f_n(t) := \Big(\langle f(t), e_1\rangle, \dots, \langle f(t), e_n\rangle\Big), \qquad t \in]0,T[,$$
$$y_{0,n} := \Big((u_{0,n}, e_1)_V, \dots, (u_{0,n}, e_n)_V\Big).$$

Then we want y_n to solve the linear Cauchy problem in $\mathbb{R}^n$

$$B_n y_n'(s) + A_n y_n(s) = f_n(s) \qquad \text{for a.a. } s \in]0,T[, \tag{2.23}$$

$$y_n(0) = y_{0,n}, \tag{2.24}$$

where $y_n(s)$, $f_n(s)$, and $y_{0,n}$ are meant as column vectors. Let us note that $f_n \in L^1(0,T;\mathbb{R}^n)$, in fact if $f = f_{V'} + f_H$, with $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^1(0,T;H)$, then we have that for $i = 1,\dots,n$ and for almost all $t \in]0,T[$

$$\begin{aligned} |\langle f(t), e_i\rangle| &= |\langle f_{V'}(t) + f_H(t), e_i\rangle| = |\langle f_{V'}(t), e_i\rangle + (f_H(t), e_i)| \\ &\le \|f_{V'}(t)\|_{V'}\|e_i\|_V + \|f_H(t)\|_H\|e_i\|_H. \end{aligned}$$

As $e_1,\dots,e_n$ are linearly independent and $(\cdot,\cdot)$ is a scalar product on V_n, we have that B_n is invertible, thus thanks to standard theorems for ordinary differential equations, there exists a unique $y_n \in W^{1,1}(0,T;\mathbb{R}^n)$ satisfying (2.23)–(2.24) (for example we can apply Cauchy-Lipschitz Theorem to the equation $y_n' = B_n^{-1}f_n - B_n^{-1}A_n y_n$). This means that there is one and only one function $u_n \in W^{1,1}(0,T;V_n)$ such that (2.21)–(2.22) hold, therefore the proposition is proved. Incidentally notice that a comparison in equation (2.23) yields $u_n' \in L^2(0,T;V_n) + L^1(0,T;V_n)$ because it is easily seen that f_n belongs to the same space; however this is immaterial for our purpose. □

Now we would like to take the limit of u_n as $n \to \infty$ in some suitable space and to prove that the limit satisfies the original Problem **(H)**. To do this we need uniform estimates on the solution in order to establishes some weak convergences. Here is the main estimate.

Proposition I.2.2. *For any $n \in \mathbb{N}_*$ let u_n be the solution to Problem* $(\mathbf{H}_n)$. *Then there exists a constant $C > 0$, depending only on T, such that*

$$\|u_n\|_{L^\infty(0,T;H)} + \|u_n\|_{L^2(0,T;V)} \leq C \tag{2.25}$$

for all $n \in \mathbb{N}_*$.

Proof. Let us take $t \in [0, T]$ and write equation (2.19) with $v = u_n(s)$. Then let us integrate this equation over $]0, t[$. We get, thanks to Corollary A.2.1 and to (2.20),

$$\frac{1}{2}\|u_n(t)\|_H^2 + \|\nabla u_n\|^2_{L^2(0,t;\mathbf{H})} = \frac{1}{2}\|u_{0,n}\|_H^2 + \int_0^t \langle f(s), u_n(s)\rangle ds. \tag{2.26}$$

Now let us estimate the integral on the right hand side of (2.26). Let $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^1(0,T;H)$ such that $f = f_{V'} + f_H$. We have, using also Hölder and Young inequalities,

$$\begin{aligned}
\int_0^t \langle f(s), u_n(s)\rangle ds &= \int_0^t \Big(\langle f_{V'}(s), u_n(s)\rangle + (f_H(s), u_n(s))\Big) ds \\
&\leq \int_0^t \|f_{V'}(s)\|_{V'}\|u_n(s)\|_V ds + \int_0^t \|f_H(s)\|_H\|u_n(s)\|_H ds \\
&= \int_0^t \|f_{V'}(s)\|_{V'}\Big(\|u_n(s)\|_H^2 + \|\nabla u_n(s)\|_{\mathbf{H}}^2\Big)^{1/2} ds \\
&\quad + \int_0^t \|f_H(s)\|_H\|u_n(s)\|_H ds \\
&\leq \frac{1}{2}\left(\|\nabla u_n\|^2_{L^2(0,t;\mathbf{H})} + \|u_n\|^2_{L^2(0,t;H)}\right) + \frac{1}{2}\|f_{V'}\|^2_{L^2(0,t;V')} \\
&\quad + \int_0^t \|f_H(s)\|_H\|u_n(s)\|_H ds.
\end{aligned} \tag{2.27}$$

Therefore, from (2.17) and (2.26) we infer that there is a constant $C > 0$, depending only on $\|u_0\|_H$, such that

$$\begin{aligned}
&\frac{1}{2}\|u_n(t)\|_H^2 + \frac{1}{2}\|\nabla u_n\|^2_{L^2(0,t;\mathbf{H})} \\
&\leq \frac{1}{2}\left(C + \|f_{V'}\|^2_{L^2(0,t;V')}\right) + \int_0^t \|u_n(s)\|_H^2 ds + \int_0^t \|f_H(s)\|_H\|u_n(s)\|_H ds.
\end{aligned}$$

Hence an application of an extended form of the Gronwall Lemma (see Proposition A.5.2) let us infer that there exists a constant $C > 0$, depending only on T, $\|u_0\|_H$, and $\|f\|_{L^2(0,T;V')+L^1(0,T;H)}$, such that

$$\|u_n\|^2_{L^\infty(0,T;H)} + \|\nabla u_n\|^2_{L^2(0,T;\mathbf{H})} \leq C. \tag{2.28}$$

Hence the continuous embedding of $L^2(0,T;H)$ in $L^\infty(0,T;H)$ yields estimate (2.25). $\square$

The estimate obtained in Proposition I.2.2 is sufficient to our aims. Hence we can state and prove the theorem about existence of weak solutions for the heat equation.

Theorem I.2.1. *Problem* **(H)** *has a unique solution.*

Proof. We divide the proof in two steps.

Uniqueness

Let $t \in [0,T]$ and take $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^1(0,T;H)$ such that $f = f_{V'} + f_H$. Let us apply equation (2.14) to u and integrate in time from 0 to t. Then, thanks to (2.10) and arguing as in (2.27), we find

$$\begin{aligned} &\frac{1}{2}\|u(t)\|^2_H + \frac{1}{2}\|\nabla u\|^2_{L^2(0,t;\mathbf{H})} \\ &\leq \frac{1}{2}\left(\|u_0\|^2_H + \|f_{V'}\|^2_{L^2(0,t;V')}\right) + \int_0^t \|u(s)\|^2_H ds \\ &\quad + \int_0^t \|f_H(s)\|_H \|u(s)\|_H ds. \end{aligned}$$

Hence we can apply the generalized version of Gronwall Lemma and deduce that there exists a constant $C > 0$, depending only on T, such that

$$\|u\|^2_{L^\infty(0,T;H)} + \|\nabla u\|^2_{L^2(0,T;\mathbf{H})} \leq C\left(\|u_0\|^2_H + \|f\|^2_{L^2(0,T;V')+L^1(0,T;H)}\right). \tag{2.29}$$

This inequality clerly entails that Problem **(H)** has at most one solution.

Existence

Proposition I.2.2 and the properties of weak and weak star compactness of the balls in $L^2(0,T;V)$ and in $L^\infty(0,T;H)$, allows us to find a function $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and to extract a subsequence, which we still denote by (u_n), such that

$$u_n \rightharpoonup u \quad \text{in } L^2(0,T;V), \tag{2.30}$$

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H) \tag{2.31}$$

as $n \to \infty$. Moreover, as $A \in \mathscr{L}(V, V')$, we also deduce from (2.30) that

$$Au_n \rightharpoonup Au_n \quad \text{in } L^2(0,T;V'). \tag{2.32}$$

We will see now that these convergences allow us to deduce that u solves Problem **(H)**. The passage to the limit, although simple, is not straightforward because we are not able to recover a bound of u_n' in some convenient space. [1] First of all let us observe that, if u were a solution of Problem **(H)**, then by Theorem A.2.2, the map $t \mapsto \langle u'(t), v\rangle$, $v \in V$ fixed, is equal to the distributional derivative of $\langle u(\cdot), v\rangle = (u(\cdot), v)$. Hence (2.16) is equivalent to

$$-\int_0^T \varphi'(t)(u(t), v)dt + \int_0^T \varphi(t)\langle Au(t), v\rangle dt = \int_0^T \varphi(t)\langle f(t), v\rangle dt \qquad \forall \varphi \in C_c^\infty(0,T), \quad \forall v \in V. \tag{2.33}$$

Then let us consider an arbitrary function $\varphi \in C_c^\infty(0,T)$ and fix $n \in \mathbb{N}_*$. Let us multiply equation (2.19) by φ and integrate in time. The formula of integration by parts gives

$$-\int_0^T \varphi'(t)(u_n(t), v)dt + \int_0^T \varphi(t)\langle Au_n(t), v\rangle dt = \int_0^T \varphi(t)\langle f(t), v\rangle dt \qquad \forall v \in V_n.$$

Moreover, since the sequence of subspaces (V_n) is increasing, we also have

$$-\int_0^T (u_n(t), \varphi'(t)v)dt + \int_0^T \langle Au_n(t), \varphi(t)v\rangle dt = \int_0^T \langle f(t), \varphi(t)v\rangle dt \qquad \forall v \in V_m, \quad \forall n \geq m. \tag{2.34}$$

As $\varphi(\cdot)v \in C_c^\infty(0,T;V_m) \subseteq L^2(0,T;V)$, thanks to (2.30)–(2.32) we can take the limit in (2.34) as $n \to \infty$ (m fixed) and we obtain

$$-\int_0^T (u(t), \varphi'(t)v)dt + \int_0^T \langle Au(t), \varphi(t)v\rangle dt = \int_0^T \langle f(t), \varphi(t)v\rangle dt \qquad \forall v \in V_m \tag{2.35}$$

for all $m \in \mathbb{N}_*$. Hence we deduce that formula (2.35) holds also with V_m replaced by V_∞. Finally by density we obtain (2.33). In fact any $v \in V$ is the limit in V of a sequence (v_n) of elements in V_∞, so that $\varphi v_n \to \varphi v$ and $\varphi' v_n \to \varphi' v$ in

[1] See however, e.g., [18, p. 354] or [23, p. 75], where a suitable choice of the Hilbert basis of V allows to obtain this bound. But the procedures used in these two references exploit the compactness of V in H, whereas we did not utilize any compactness argument.

$L^2(0,T;V)$, and this allows us to pass to the limit in equation (2.35) written for v_n, and infer (2.33) after an integration by parts. Therefore (2.12) and (2.14) are proved, and (2.13) follows from a comparison in the equation.

It remains to prove the initial condition (2.15). We will exploit again the tecnique of multiplying the approximated equation by a suitable test function. In order to get information on $u(0)$, we consider a function $\psi \in C^\infty([0,T])$ such that $\psi(0) \neq 0$. Let us multiply (2.19) by ψ. Integrating by parts, thanks to (2.22), we get

$$\begin{aligned} &-\int_0^T \psi'(t)(u_n(t),v)dt + \int_0^T \psi(t)\langle Au_n(t),v\rangle dt \\ &= \psi(0)(u_{0,n},v) + \int_0^T \psi(t)\langle f(t),v\rangle dt \qquad \forall v \in V. \end{aligned}$$

Using the same procedure as above, since $u_{0,n} \to u_0$ in V, we finally find

$$\begin{aligned} &-\int_0^T \psi'(t)(u(t),v)dt + \int_0^T \psi(t)\langle Au(t),v\rangle dt \\ &= \psi(0)(u_0,v) + \int_0^T \psi(t)\langle f(t),v\rangle dt \qquad \forall v \in V. \end{aligned}$$

Now let us multiply (2.16) by ψ and integrate in time. Using again integration parts, this time we get

$$\begin{aligned} &-\int_0^T \psi'(t)(u(t),v)dt + \int_0^T \psi(t)\langle Au(t),v\rangle dt \\ &= \psi(0)(u(0),v) + \int_0^T \psi(t)\langle f(t),v\rangle dt \qquad \forall v \in V. \end{aligned}$$

Hence comparing the two previous equations we deduce that

$$\psi(0)(u_0,v) = \psi(0)(u(0),v) \qquad \forall v \in V,$$

and therefore, since $\psi(0) \neq 0$, we have that $(u_0 - u(0), v) = 0$ for any $v \in V$. Thus, V being dense in H, we have found the initial condition (2.15). □

As Problem **(H)** is linear, arguing exactly as in the uniqueness-part of the proof of Theorem I.2.1 we get the continuous dependence of solutions with respect to data, i.e. the following proposition holds.

Proposition I.2.3. *Let $f_i \in L^2(0,T;V') + L^1(0,T;H)$ and $u_{0i} \in H$, $i = 1,2$. Moreover let u_i be the solution of Problem* **(H)** *with f and u_0 replaced respectively by f_i and u_{0i}, $i = 1,2$. Then there exists a constant $C > 0$, depending only on T, such that*

$$\begin{aligned} &\|u_1 - u_2\|_{L^2(0,T;V)} + \|u_1 - u_2\|_{L^\infty(0,T;H)} \\ &\leq C\left(\|u_{01} - u_{02}\|_H + \|f_1 - f_2\|_{L^2(0,T;V')+L^1(0,T;H)}\right). \end{aligned}$$

I.3 Monotone graphs in $\mathbb{R}^2$

As the Stefan problem involves the multivalued graph sign, in this section we recall and study some properties of a special class of multivalued functions from $\mathbb{R}$ into itself. We also show some heuristic arguments in order to motivate the method used in the sequel to solve the Stefan problem.

A multivalued function from $\mathbb{R}$ into itself, i.e. a map $\alpha : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$, will be called *multivalued operator* or more simply *operator* (*in* $\mathbb{R}$). The term "operator" is perhaps not adequate in dealing with real multifunctions of real variables, but we use it to be consistent with the customary terminology adopted in the framework of Banach spaces. Indeed every definition we are going to recall has an analogous notion when $\mathbb{R}$ is substituted with a general Hilbert or Banach space (cf. [5], [4], or [33]). Any operator α can be identified with the subset of $\mathbb{R}^2$ given by its *reduced graph* $G_{\mathcal{R}}(\alpha) := \{(r,s) \in \mathbb{R}^2 \ : \ s \in \alpha(r)\}$. The *domain* of α is the set $D(\alpha) := \{r \in \mathbb{R} \ : \ \alpha(r) \neq \varnothing\}$ and its *reduced image* is the set $\mathrm{Im}_{\mathcal{R}}(\alpha) := \bigcup_{r\in\mathbb{R}} \alpha(r)$. We call *inverse operator* of α the operator $\alpha^{-1} : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ defined by the relation

$$s \in \alpha^{-1}(r) \quad \overset{def}{\Longleftrightarrow} \quad r \in \alpha(s).$$

The sum of two operators α and β and the product by a scalar $\lambda \in \mathbb{R}$ have the obvious definitions $(\alpha + \beta)(r) := \alpha(r) + \beta(r)$ and $(\lambda\alpha)(r) := \lambda\alpha(r)$ for $r \in \mathbb{R}$. We have $D(\alpha + \beta) = D(\alpha) \cap D(\beta)$ and $D(\lambda\alpha) = D(\alpha)$.

Definition I.3.1. Let $\alpha : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ be an operator. We say that α is a *monotone operator* if

$$(s_1 - s_2)(r_1 - r_2) \geq 0 \quad \forall r_i \in \mathbb{R}, \quad \forall s_i \in \alpha(r_i),\ i = 1,2.$$

We say that α is *maximal monotone* if α is monotone and if its reduced graph is maximal in the set of monotone operators, ordered with the inclusion of sets.

Equivalently α is maximal monotone if and only if it is monotone and if the following condition holds:

$$\Big[a, b \in \mathbb{R},\ (b-s)(a-r) \geq 0 \qquad \forall r \in D(\alpha),\ \forall s \in \alpha(r)\Big] \quad \Longrightarrow \quad b \in \alpha(a). \quad (3.1)$$

If α is maximal monotone, using (3.1) it is straightforward to see that $\alpha(r)$ is a closed and convex set for any $r \in \mathbb{R}$, thus we call *minimal section* of α the increasing function $\alpha_0 : D(\alpha) \longrightarrow \mathbb{R}$ by defining $\alpha_0(r)$ to be the unique number s_0 such that $|s_0| = \min |\alpha(r)|$. $\diamondsuit$

Of course here "monotone" means "increasing", that would be a more appropriate term, however we continue to adopt the standard terminology. Maximality of a monotone operator means that it is, in some sense, continuous, and in fact it can be proved that its reduced graph is a one-dimensional Lipschitz submanifold of $\mathbb{R}^2$ without boundary (see [27] and [2]). Notice that, given $\lambda > 0$, α is (maximal) monotone if and only if α^{-1} is (maximal) monotone if and only if $\lambda\alpha$ is (maximal) monotone. It is also easy to verify that if α is maximal monotone, then $D(\alpha)$ is a convex set and $\alpha(r) = [\alpha_0(r-), \alpha_0(r+)]$ for any r (the domain may fail to be convex in higher dimensions). Finally let us note that if α is maximal monotone and $D(\alpha) \neq \mathbb{R}$ then $\mathrm{Im}_{\mathcal{R}}(\alpha + \mathrm{Id}_{\mathbb{R}}) = \mathbb{R}$.

In the following proposition we show some properties of maximal monotone operators which will be useful in the sequel.

Proposition I.3.1. *Let $\alpha : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ be monotone. Then $(\alpha + \mathrm{Id}_{\mathbb{R}})^{-1} : \mathrm{Im}_{\mathcal{R}}(\alpha + \mathrm{Id}_{\mathbb{R}}) \longrightarrow \mathscr{P}(\mathbb{R})$ is single-valued and it is a Lipschitz function with $\mathrm{Lip}((\alpha + \mathrm{Id}_{\mathbb{R}})^{-1}) = 1$. If in addition α is maximal, then $\mathrm{Im}_{\mathcal{R}}(\alpha + \mathrm{Id}_{\mathbb{R}}) = \mathbb{R}$.*

Proof. By monotonicity we have

$$|(r_1 - r_2) + (s_1 - s_2)|^2 \geq |r_1 - r_2|^2 \qquad \forall r_i \in D(\alpha), \quad \forall s_i \in \alpha(r_i),\ i = 1, 2,$$

therefore if $r_1 + s_1 = r_2 + s_2$ then $r_1 = r_2$ and this proves that $(\alpha + \mathrm{Id}_{\mathbb{R}})^{-1}$ is single-valued. Moreover the above inequality shows that the map is nonexpansive and the fact that $(\alpha + \mathrm{Id}_{\mathbb{R}})^{-1}(r_i + s_i) = r_i$ yields the value 1 for the Lipschitz constant. To prove the last part of the statement, we can argue by contradiction assuming that there exists $s_0 \in \mathbb{R}$ such that $s_0 \notin \mathrm{Im}_{\mathcal{R}}(\alpha + \mathrm{Id}_{\mathbb{R}})$. Since the function $\alpha_0 + \mathrm{Id}_{\mathbb{R}}$ is increasing and onto from $\mathbb{R}$ in itself, there is a number r_0 such that $s_0 \in [\alpha_0(r_0-) + r_0, \alpha_0(r_0+) + r_0]$. At this point it is easy to contradict the maximality of α by constructing the monotone monotone map $\widetilde{\alpha}$ whose graph is $G_{\mathcal{R}}(\widetilde{\alpha}) := G_{\mathcal{R}}(\alpha) \cup \{(r_0, s_0 - r_0)\}$. $\square$

Now let us consider the Stefan problem introduced in the first section,

$$\frac{\partial(\theta+\chi)}{\partial t} - \Delta\theta = f, \tag{3.2}$$

$$\chi \in \operatorname{sign}(\theta), \tag{3.3}$$

and we couple this problem with suitable initial and boundary conditions that now we do not specify (a possible choice is given in (1.19)–(1.20)). Similarly to the case of the heat equation, also in this case we would like to approximate Problem (3.2)–(3.3), to reduce to a problem which we are able to solve, for instance by means of a fixed point technique and with the aid of the theorems about heat equation proved in the previous section. A possible strategy could be to approximate the maximal monotone operator sign for example by slightly tilting the vertical part of its graph in order to obtain a singlevalued function $\operatorname{sign}_\lambda$ with a slope of $1/\lambda$ for some small $\lambda > 0$. To be more precise we could define

$$\operatorname{sign}_\lambda(r) := \begin{cases} -1 & \text{if } r < -\lambda \\ r/\lambda & \text{if } |r| \le \lambda \\ 1 & \text{if } r > -\lambda \end{cases}.$$

Thus, making λ go to zero, we have that the tilted graph "tends" to the graph of the sign. Observe that $\operatorname{sign}_\lambda$ is Lipschitz continuous. In this way the inclusion (3.3) is approximated by the equality

$$\chi = \operatorname{sign}_\lambda(\theta). \tag{3.4}$$

Then we rewrite equation (3.2) in the following way

$$\frac{\partial\theta}{\partial t} - \Delta\theta = f - \frac{\partial\chi}{\partial t}, \tag{3.5}$$

and we look for a fixed point of the map $\Theta \mapsto \theta$ which to any function Θ associates the solution of equation $\partial_t\theta - \Delta\theta = f - \partial_t\chi$, with $\chi = \operatorname{sign}_\lambda(\Theta)$. In order to solve equation (3.5) (with χ fixed) we can exploit the results of the previous section, and the fixed point of the map $\Theta \mapsto \theta$ is therefore a solution of the approximated problem (3.2), (3.4). Moreover observe also that in the approximation the phase function should be more regular than in original problem, indeed it is clear from Section I.1 that in general $\partial_t\chi$ does not exists. Finally we would like to make λ go to zero and to prove that the in the limit the original relations (3.2)–(3.3) are satisfied. The hardest part of this tentative procedure is the passage to the limit in the nonlinear equation $\chi = \operatorname{sign}_\lambda(\theta)$. To make this limit procedure successful we need enough *a priori* estimates of the solutions in some suitable spaces. The formal estimate we are going to perform could also suggest the "right" way to approximate the original problem (there could be more than one). The key point to obtain enough a priori bounds is to rewrite inclusion (3.3) in the equivalent way

$$\operatorname{sign}^{-1}(\chi) \ni \theta, \tag{3.6}$$

accordingly we will approximate sign^{-1} rather than the sign operator. We use again the idea of tilting the vertical parts of its graph, in order to obtain a monotone function whose graph is quite "near" to the original graph. Therefore for $\lambda > 0$ we define $(\operatorname{sign}^{-1})_\lambda$ by

$$(\operatorname{sign}^{-1})_\lambda(r) := \begin{cases} (r+1)/\lambda & \text{if } r < -1 \\ 0 & \text{if } |r| \leq 1 \\ (r-1)/\lambda & \text{if } r > 1 \end{cases}.$$

The reason why we invert the inclusion (3.3) will be clear in a moment, but admittedly there is much hindsight in our argument.

Now we perform the a priori estimates directly on equations (3.2)–(3.3). All the arguments employed will be only formal and the all functions will be supposed very smooth to legitimate all the calculations. For simplicity we will also assume that $f \equiv 0$ and that θ is required to be zero on the boundary. The first step is to recover some kind of estimate for the temperature θ. In order to do this we multiply equation (3.2) by θ and integrate over the cylinder $\Omega\times]0,t[$, $t \in [0,T]$. Integration by parts in time and an application of Green formula to evaluate $\int_0^t\int_\Omega (\Delta\theta)\theta$ gives

$$\begin{aligned} \frac{1}{2}\|\theta(\cdot,t)\|^2_{L^2(\Omega)} &+ \int_0^t\int_\Omega \theta(x,s)\frac{\partial\chi}{\partial t}(x,s)dxds \\ &+ \|\nabla\theta(\cdot,t)\|^2_{L^2(\Omega;\mathbb{R}^d)} = \frac{1}{2}\|\theta(\cdot,0)\|^2_{L^2(\Omega)} \end{aligned} \tag{3.7}$$

(recall that we are assuming $\theta|_{\partial\Omega} \equiv 0$). The problem is now to estimate the integral at the left hand side of (3.7). This estimate can be done by multiplying (3.6) by $\partial_t\chi$, and this explains why we inverted the inclusion (3.3). The heuristic argument is the following. Let us perform such multiplication assuming that sign^{-1} is more regular, still increasing, but without vertical parts in its graph. We can argue for example on the approximation given by $(\operatorname{sign}^{-1})_\lambda$. If we integrate in time and space, we obtain

$$\int_0^t\int_\Omega \theta(x,s)\frac{\partial\chi}{\partial t}(x,s)dxds = \int_0^t\int_\Omega (\operatorname{sign}^{-1})_\lambda(\chi(x,s))\frac{\partial\chi}{\partial t}(x,s)dxds, \tag{3.8}$$

then we are reduced to control the integral at the right hand side. Denote by p_λ the primitive of $(\operatorname{sign}^{-1})_\lambda$ crossing the origin. Then we have that $p_\lambda \geq 0$ because

$(\text{sign}^{-1})_\lambda$ is increasing and null in zero. Hence in this regular setting we can write

$$\int_0^t \int_\Omega (\text{sign}^{-1})_\lambda(\chi(x,s)) \frac{\partial \chi}{\partial t}(x,s) dx ds = \int_0^t \int_\Omega \frac{d}{ds} p_\lambda(\chi(x,s)) dx ds$$
$$= \int_\Omega p_\lambda(\chi(x,t)) dx - \int_\Omega p_\lambda(\chi(x,0)) dx \geq - \int_\Omega p_\lambda(\chi(x,0)) dx \tag{3.9}$$

(here we need to require $p_\lambda(\chi(\cdot,0)) \in L^1(\Omega)$). Since in the Stefan model the relation (3.3) is required to hold, we may assume that the system starts evolving from a initial state such that the inclusion $\chi(\cdot,0) \in \text{sign}(\theta(\cdot,0))$ makes sense, i.e. we assume that $\chi(x,0) \in [-1,1]$ for all x. Consequently $p_\lambda(\chi(x,0)) = 0$ ($p_\lambda = 0$ on $[-1,1]$) and from (3.8) and (3.9) we infer that

$$\int_0^t \int_\Omega \theta(x,s) \frac{\partial \chi}{\partial t}(x,s) dx ds \geq -\|p_\lambda(\chi(\cdot,0))\|_{L^1(\Omega)} = 0. \tag{3.10}$$

Now we can add (3.10) and (3.7), and we finally get the estimate

$$\frac{1}{2}\|\theta(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla\theta(\cdot,t)\|^2_{L^2(\Omega;\mathbb{R}^d)} \leq C \tag{3.11}$$

for some positive constant C depending only on the initial data. Moreover observe that χ remains bounded between -1 and 1, so that we have a bound for its $L^\infty(Q)$ norm.

All the estimates we have formally obtained turn out to be sufficient to pass to the limit and to get a solution of the Stefan problem. We will not further continue in describing a heuristic argument to show that the limit procedure works, in fact this will be clear after the rigorous proofs will be done.

The point we want to clarify instead is what kind of approximated problem we have to study. Regarding the regularization of the multivalued map we have already a proposal. Now observe that the formal estimates that we performed need some time-regularity of the phase function χ: we want $\partial_t \chi$ to exists and to be at least integrable. A common technique in PDEs consists in *perturbing* the equations by adding a term that makes solutions more regular. We explain this technique in our specific case. If we add the term $\varepsilon\partial_t\chi$ to the left hand side of (3.6) we get the equation (inclusion)

$$\varepsilon \frac{\partial \chi}{\partial t} + \text{sign}^{-1}(\chi) \ni \theta. \tag{3.12}$$

In this way the solution of our problem is expected to be more regular, in such a way that the previous a priori estimates can be actually performed. Then one passes to the limit for $\varepsilon \searrow 0$ and tries to prove that the non-perturbed equation is satisfied.

The previous heuristic discussion aims to show that, in order to solve (3.2)–(3.3), a possible regularization could be given by

$$\frac{\partial(\theta+\chi)}{\partial t} - \Delta\theta = f, \tag{3.13}$$

$$\varepsilon\frac{\partial\chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta. \tag{3.14}$$

In the second relation (3.14) we did not write $(\operatorname{sign}^{-1})_\lambda$ instead of $(\operatorname{sign}^{-1})$ because the inclusion (3.14) has its own interest and more generally a problem like

$$\frac{\partial u}{\partial t} + \alpha(u) \ni F \tag{3.15}$$

can be studied, with α maximal monotone operator in $\mathbb{R}$, F given, and u unknown. Moreover we want also point out that problem (3.13)–(3.14) has a precise physical meaning that is strictly related to the Stefan problem. This problem is usually called *Stefan problem with phase relaxation* or more simply *relaxed Stefan problem*, and its physical meaning will be explained in Chapter II.

In order to treat with any maximal monotone mapping, now we give a general procedure to approximate an arbitrary maximal monotone graph with a graph of a Lipschitz continuous increasing function. As a particular case, we will obtain the approximations of sign and sign^{-1} given before. If α is the operator we would like to regularize, the procedure consists in performing a shear of its graph in the x-direction with a factor $\lambda > 0$, in other terms we transform the graph of α under the mapping $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T_\lambda(r,s) := (r+\lambda s, s)$, $(r,s) \in \mathbb{R}^2$. Hence we give the following precise definition.

Definition I.3.2. Let $\alpha : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ be a maximal monotone operator. We call λ-*approximation* of α the operator α_λ whose reduced graph is given by

$$G_{\mathcal{R}}(\alpha_\lambda) := \{(r+\lambda s, s) \ : \ s \in \alpha(r)\}, \tag{3.16}$$

i.e.

$$\alpha_\lambda := \left(\alpha^{-1} + \lambda \mathrm{Id}_{\mathbb{R}}\right)^{-1}.$$

The map α_λ is also called *Yosida approximation.* Moreover we call λ-*resolvent* of α the operator $j_\lambda := (\mathrm{Id}_{\mathbb{R}} + \lambda\alpha)^{-1}$. $\diamondsuit$

In the next proposition there are collected some properties of the Yosida approximation and of the resolvent.

Proposition I.3.2. *Let $\alpha : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ be a maximal monotone operator and let $\lambda > 0$. Then*

(i) α_λ *is an increasing single-valued Lipschitz function, $D(\alpha_\lambda) = \mathbb{R}$, and* $\mathrm{Lip}(\alpha_\lambda) = 1/\lambda$.

(ii) j_λ *is a single-valued Lipschitz function, $D(j_\lambda) = \mathbb{R}$, and* $\mathrm{Lip}(j_\lambda) = 1$. *Moreover $\alpha_\lambda(r) \in \alpha(j_\lambda(r))$ and $r = \lambda\alpha_\lambda(r) + j_\lambda(r)$ for all $r \in \mathbb{R}$.*

(iii) *For each $r \in D(\alpha)$ we have $|\alpha_\lambda(r)| \leq |\alpha_0(r)|$ for all $\lambda > 0$ and $\alpha_\lambda(r) \to \alpha_0(r)$ as $\lambda \searrow 0$.*

Proof.

(i)
It suffices to observe that $\alpha_\lambda = \frac{1}{\lambda}(\frac{1}{\lambda}\alpha^{-1} + \mathrm{Id}_\mathbb{R})^{-1}$ and to apply Proposition I.3.1.

(ii)
The first part about Lipschitz continuity is proved as in the point (i). The other assertions are straightforward consequences of the definition of j_λ.

(iii)
As $\alpha_\lambda \in \alpha(j_\lambda(r))$ for all λ, we have $0 \leq (\alpha_0(r) - \alpha_\lambda(r))(r - j_\lambda(r)) = (\alpha_0(r) - \alpha_\lambda(r))(\lambda\alpha_\lambda(r)) = \lambda\alpha_\lambda(r)\alpha_0(r) - (\alpha_\lambda(r))^2$, hence $|\alpha_\lambda(r)| \leq |\alpha_0(r)|$. Therefore the sequence $(\alpha_\lambda(r))$ is bounded and from point (ii) we infer that

$$\lim_{\lambda \searrow 0} j_\lambda(r) = r. \tag{3.17}$$

Now $\alpha_\lambda(r) \in \alpha(j_\lambda(r)) = [\alpha_0(j_\lambda(r)-), \alpha_0(j_\lambda(r)+)]$, therefore, since α_0 is increasing, we deduce from (3.17) that $\alpha_\lambda(r) \to \alpha_0(r)$. □

An important feature of maximal monotone operators in $\mathbb{R}$ is that they admit a "primitive" function. To be more precise, given α maximal monotone, there exists a function φ, that is convex and lowersemicontinuous, such that its subdifferential $\partial\varphi$ is exactly α. This property does not hold for higher dimensions. We recall here briefly the notion of subdifferential for convex functions on $\mathbb{R}$.

Definition I.3.3. Let $\varphi : \mathbb{R} \longrightarrow \widetilde{\mathbb{R}}$ be proper and convex and let $r_0 \in \operatorname{dom}\varphi := \{r \ : \ \varphi(r) \neq \infty\}$. The *subdifferential* of φ at r_0 is the set

$$\partial\varphi(r_0) := \{s_0 \in \mathbb{R} \ : \ s_0(r - r_0) + \varphi(r_0) \leq \varphi(r) \ \forall r \in \mathbb{R}\}.$$

Notice that $\partial\varphi(r_0) = [\varphi'(r_0-), \varphi'(r_0+)]$. Eventually we set $\partial\varphi(r_0) = \varnothing$ if $r_0 \notin \operatorname{dom}\varphi$, thus we have defined an operator $\partial\varphi : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ which we still call *subdifferential* of φ. ♢

It is easy to prove the following

Proposition I.3.3. *If $\varphi : \mathbb{R} \longrightarrow \widetilde{\mathbb{R}}$ is a proper, convex and lowersemicontinuous function then its subdifferential $\partial\varphi$ is a maximal monotone operator.*

We will not use the previous proposition, we need instead some properties about primitives of monotone operators we mentioned before. Here they are.

Proposition I.3.4. *Let α be a maximal monotone operator in $\mathbb{R}$. Let $r_0 \in D(\alpha)$ and let $\varphi : \mathbb{R} \longrightarrow \widetilde{\mathbb{R}}$ be defined by*

$$\varphi(r) := \begin{cases} \displaystyle\int_{r_0}^{r} \alpha_0(s)ds & \text{if } r \in \overline{D(\alpha)} \\ \infty & \text{if } r \notin \overline{D(\alpha)} \end{cases}. \tag{3.18}$$

For all $\lambda > 0$ define $\varphi_\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\varphi_\lambda(r) := \int_{r_0}^{r} \alpha_\lambda(s)ds, \qquad r \in \mathbb{R}. \tag{3.19}$$

Then φ is a lowersemicontinuous and convex function such that $\operatorname{dom}\varphi \subseteq \overline{D(\alpha)}$ and $\partial\varphi = \alpha$. Moreover φ_λ is convex and differentiable for any $\lambda > 0$, $\varphi'_\lambda = \alpha_\lambda$, and $\varphi_\lambda(r)$ converges upward to $\varphi(r)$ for each $r \in \mathbb{R}$.

Proof. It easy to see that φ and φ_λ satisfies the required regularity properties and that $\partial\varphi = \alpha$ and $\varphi'_\lambda = \alpha_\lambda$. The convergence of φ_λ is readily proved by exploiting Proposition (I.3.2)-(iii) and the Lebesgue Dominated Convergence Theorem. □

Now we have at disposal all the ingredients to study and solve the equation (3.15).

Theorem I.3.1. *Let Ω a bounded open set in $\mathbb{R}^d$ ($d \in \mathbb{N}_*$) and let $T > 0$. Assume that $F \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and α is a maximal monotone operator in $\mathbb{R}$. Define $\varphi : \mathbb{R} \longrightarrow \widetilde{\mathbb{R}}$ as in (3.18), where $r_0 \in D(\alpha)$ is arbitrarily fixed. If we assume that $\varphi \circ u_0 \in L^1(\Omega)$ then there exists a unique $u \in H^1(0,T;L^2(\Omega))$ such that $u \in D(\alpha)$ a.e. in Q and*

$$u' + \alpha(u) \ni F \qquad \text{a.e. in } Q := \Omega\times]0,T[, \tag{3.20}$$

$$u(0) = u_0. \tag{3.21}$$

Proof. To shorten notations we set $H := L^2(\Omega)$. We prove the theorem in several steps.

Uniqueness

This is straighforward. Indeed if u_1 and u_2 are two solutions, multiplying by $u_1 - u_2$ the difference of the respective equations (3.20), integrating in time and space, and using the monotonicity of α, we get $u_1 = u_2$ almost everywhere in Q.

Approximation

For any $\lambda > 0$ let α_λ be the Yosida approximation of α. By the Cauchy-Lipschitz-Picard Theorem there exists one and only one function $u_\lambda \in W^{1,1}(0,T;H)$ such that

$$u'_\lambda(t) + \alpha_\lambda(u_\lambda(t)) = F \qquad \forall t \in [0,T], \tag{3.22}$$

$$u_\lambda(0) = u_0. \tag{3.23}$$

By comparison in this equation we also infer that $u_\lambda \in H^1(0,T;H)$. Now let φ_λ the function defined in (3.19) and define $\psi_\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ setting

$$\psi_\lambda(r) := \varphi_\lambda(r) - \alpha_\lambda(r_0)(r - r_0), \qquad r \in \mathbb{R}. \tag{3.24}$$

Observe that $\psi_\lambda(r_0) = 0$ and

$$\psi'_\lambda = \varphi'_\lambda - \alpha_\lambda(r_0) = \alpha_\lambda - \alpha_\lambda(r_0). \tag{3.25}$$

Moreover, thanks to Proposition (I.3.2)-(iii) we have that

$$0 \le \psi_\lambda(r) \le |\varphi(r)| + |\alpha_0(r_0)||r - r_0| \qquad \forall r \in \mathbb{R}, \tag{3.26}$$

so that $\psi_\lambda(u_0) \in L^1(\Omega)$.

Energy estimate

Multiply equation (3.22) by u'_λ and integrate over $\Omega\times]0,t[$, with $t \in [0,T]$. Taking (3.25) into account, we get

$$\begin{aligned}\|u'_\lambda\|^2_{L^2(0,t;H)} &+ \int_0^t \int_\Omega \big(\varphi'_\lambda(u_\lambda(s) - \alpha_\lambda(r_0)\big)u'_\lambda(s)ds \\ &= \int_0^t \int_\Omega F(s)u'_\lambda(s)ds - \int_0^t \int_\Omega \alpha_\lambda(r_0)u'_\lambda(s)ds.\end{aligned} \tag{3.27}$$

Then, due to (3.25) and (3.23), we find

$$\frac{1}{2}\|u'_\lambda\|^2_{L^2(0,t;H)} + \int_\Omega \psi_\lambda(u_\lambda(t)) \le \int_\Omega \psi_\lambda(u_0) + \|F\|^2_{L^2(0,t;H)} + t|\Omega||\alpha_\lambda(r_0)|^2.$$

Hence, thanks to (3.26), there is a constant $C > 0$, independent of λ, but depending only on $\|\varphi(u_0)\|_{L^1(\Omega)}$, $\|u_0\|_{L^1(\Omega)}$, $|\alpha_0(r_0)|$, $\|F\|_{L^2(0,T;H)}$, T, and $\mathcal{L}^d(\Omega)$, such that

$$\|u'_\lambda\|_{L^2(0,T;H)} \le C. \tag{3.28}$$

Therefore by a comparison in (3.22) we get

$$\|\alpha_\lambda(u_\lambda)\|_{L^2(0,T;H)} \le C \tag{3.29}$$

for some positive constant C having the same dependences as above.

Cauchy estimate

Let us fix $\lambda, \mu > 0$ arbitrarily, take the difference of the respective equations (3.22) for λ and μ, and multiply it by $u_\lambda - u_\mu$. After an integration in time and space we have

$$\frac{1}{2}\|u_\lambda(t) - u_\mu(t)\|_H^2 + \int_0^t \int_\Omega \big(\alpha_\lambda(u_\lambda(s)) - \alpha_\mu(u_\mu(s))\big)\big(u_\lambda(s) - u_\mu(s)\big)ds = 0. \tag{3.30}$$

Since $\alpha_\lambda(r) \in \alpha(j_\lambda(r))$ and $r = \lambda\alpha_\lambda(r) + j_\lambda(r)$ for all $r \in \mathbb{R}$ (Proposition I.3.2-(ii)), we can write, omitting the dependence on s,

$$\begin{aligned}
\big(\alpha_\lambda(u_\lambda) - \alpha_\mu(u_\mu)\big)\big(u_\lambda - u_\mu\big) &= \big(\alpha_\lambda(u_\lambda) - \alpha_\mu(u_\mu)\big)\big(\lambda\alpha_\lambda(u_\lambda) - \mu\alpha_\mu(u_\mu)\big) \\
&\quad + \big(\alpha_\lambda(u_\lambda) - \alpha_\mu(u_\mu)\big)\big(j_\lambda(u_\lambda) - j_\mu(u_\mu)\big) \\
&\ge \big(\alpha_\lambda(u_\lambda) - \alpha_\mu(u_\mu)\big)\big(\lambda\alpha_\lambda(u_\lambda) - \mu\alpha_\mu(u_\mu)\big) \\
&= \lambda|\alpha_\lambda(u_\lambda)|^2 + \mu|\alpha_\mu(u_\mu)|^2 \\
&\quad - \lambda|\alpha_\lambda(u_\lambda)\alpha_\mu(u_\mu)| - \mu|\alpha_\lambda(u_\lambda)\alpha_\mu(u_\mu)| \\
&\ge \lambda|\alpha_\lambda(u_\lambda)|^2 + \mu|\alpha_\mu(u_\mu)|^2 \\
&\quad - \lambda\left(|\alpha_\lambda(u_\lambda)|^2 + \frac{1}{4}|\alpha_\mu(u_\mu)|^2\right) \\
&\quad - \mu\left(|\alpha_\mu(u_\mu)|^2 + \frac{1}{4}|\alpha_\lambda(u_\lambda)|^2\right) \\
&= -\frac{\lambda}{4}|\alpha_\mu(u_\mu)|^2 - \frac{\mu}{4}|\alpha_\lambda(u_\lambda)|^2.
\end{aligned}$$

Thus, exploiting also (3.29),

$$\begin{aligned}
&\int_0^t \int_\Omega \big(\alpha_\lambda(u_\lambda(s)) - \alpha_\mu(u_\mu(s))\big)\big(u_\lambda(s) - u_\mu(s)\big)ds \\
&\ge -\frac{\lambda + \mu}{4}\left(\|\alpha_\mu(u_\mu)\|_{L^2(0,t;H)}^2 + \|\alpha_\lambda(u_\lambda)\|_{L^2(0,t;H)}^2\right) \ge -C\frac{\lambda + \mu}{4}
\end{aligned}$$

with $C > 0$ independent of λ. Hence from (3.30) we deduce

$$\frac{1}{2}\|u_\lambda(t) - u_\mu(t)\|_H^2 \le C\frac{\lambda+\mu}{4} \qquad \forall t \in [0,T], \tag{3.31}$$

which entails that

$$\|u_\lambda - u_\mu\|_{L^\infty(0,T;H)} \le C(\lambda+\mu)^{1/2}.$$

Therefore (u_λ) is a Cauchy sequence in $C(0,T;H)$ and there exists a function $u \in C(0,T;H)$ such that

$$u_\lambda \to u \qquad \text{in } C([0,T];H) \tag{3.32}$$

as $\lambda \searrow 0$ (hence this sequence strongly converges in $L^2(0,T;H)$). It follows also from (3.28) that, at least for a subsequence,

$$u'_\lambda \rightharpoonup u' \qquad \text{in } L^2(0,T;H). \tag{3.33}$$

(see Proposition A.2.4). Now observe that from (3.29) we deduce that

$$\|j_\lambda(u_\lambda) - u_\lambda\|_H = \lambda\|\alpha_\lambda(u_\lambda(t))\|_H \le \lambda C \qquad \forall t \in [0,T],$$

hence

$$j_\lambda(u_\lambda) \to u \qquad \text{in } C([0,T];H) \tag{3.34}$$

as $\lambda \searrow 0$.

Limit as $\lambda \searrow 0$

First of all let us notice that $u(0) = u_0$ because (u_λ) is uniformly converging to u. It remains to prove that the inclusion (3.21) is satisfied. Recalling Proposition I.3.2-(ii), from (3.22) we deduce that

$$u'_\lambda(x,t) - F(x,t) \in \alpha_\lambda(u_\lambda(x,t)) = \alpha(j_\lambda(u_\lambda(x,t))) \qquad \text{for a.a. } (x,t) \in \Omega\times]0,T[,$$

therefore, by definition of maximal monotone operator,

$$\Big(u'_\lambda(x,t) - F(x,t) - s\Big)\Big(j_\lambda(u_\lambda(x,t)) - r\Big) \ge 0 \qquad \text{for a.a. } (x,t) \in \Omega\times]0,T[, \quad \forall r \in D(\alpha),\ \forall s \in \alpha(r).$$

Observe that the previous condition yields

$$\int_Q \Big(u'_\lambda(x,t) - F(x,t) - z(x,t)\Big)\Big(j_\lambda(u_\lambda(x,t)) - v(x,t)\Big)dxdt \ge 0 \qquad \forall v,z \in L^2(Q),\ v \in D(\alpha),\ z \in \alpha(v) \text{ a.e. in } Q. \tag{3.35}$$

Taking the limit for $\lambda \searrow 0$ in (3.35), thanks to convergences (3.33)–(3.34) we find

$$\int_Q \Big(u'(x,t) - F(x,t) - z(x,t)\Big)\Big(u(x,t) - v(x,t)\Big)dxdt \geq 0$$
$$\forall v, z \in L^2(Q),\ v \in D(\alpha),\ z \in \alpha(v) \text{ a.e. in } Q. \quad (3.36)$$

Now let us fix $\overline{v}$ and $\overline{z}$ in $L^2(Q)$ such that $\overline{v} \in D(\alpha)$ and $\overline{z} \in \alpha(\overline{v})$ a.e. in Q. Moreover let A be an arbitrary measurable subset of Q and let us write (3.36) with $v = \overline{v}\chi_A + u\chi_{\Omega \setminus A}$. We find

$$\int_A \Big(u'(x,t) - F(x,t) - \overline{z}(x,t)\Big)\Big(u(x,t) - \overline{v}(x,t)\Big)dxdt \geq 0.$$

Therefore we have that $(u' - F - \overline{z})(u - \overline{v}) \geq 0$ a.e. in Q, and this yields, by the arbitrariness of $\overline{v}$ and by (3.1), that $u' - F \in \alpha(u)$ a.e. in Q, i.e. (3.20). $\square$

Corollary I.3.1. *Under the same assumptions of Theorem I.3.1, let u be the solution of* (3.20)–(3.21)*. Then we have*

$$\|u'\|^2_{L^2(0,t;L^2(\Omega))} + \int_\Omega \big(\varphi(u(t)) - \alpha_0(r_0)(u(t) - r_0)\big)$$
$$\leq \int_\Omega \big(\varphi(u_0) - \alpha_0(r_0)(u_0 - r_0)\big) + \int_0^t\int_\Omega F(s)u'(s)ds - \int_0^t\int_\Omega \alpha_0(r_0)u'(s)ds \quad (3.37)$$

for all $t \in [0,T]$. Moreover the integral at the left hand side of (3.37) *is positive.*

Proof. Let u_λ be the solution of the regularized problem (3.22)–(3.23) considered in the proof of Theorem I.3.1. Now fix $t \in [0,T]$ and define $H := L^2(\Omega)$. Multiplying equation (3.22) by u'_λ we obtained equality (3.27), which we rewrite, using (3.25), as

$$\|u'_\lambda\|^2_{L^2(0,t;H)} + \int_\Omega \psi_\lambda(u_\lambda(t))$$
$$= \int_\Omega \psi_\lambda(u_0) + \int_0^t\int_\Omega F(s)u'_\lambda(s)ds - \int_0^t\int_\Omega \alpha_0(r_0)u'_\lambda(s)ds. \quad (3.38)$$

Now notice that owing to (3.32), we have that $u_\lambda(t) \to u(t)$ in H, then there exists a subsequence which we denote with the same symbol (u_λ), such that $u_\lambda(t)$ converges to $u(t)$ almost everywhere in Ω. On the other hand, since φ_λ is pointwise converging upward to the continuous function φ, by Dini Theorem, φ_λ

is uniformly converging on compact sets. Therefore we have $\varphi_\lambda(u_\lambda(t)) \to \varphi(u(t))$ almost everywhere in Ω and thanks to Fatou Lemma we can infer that

$$\int_\Omega \varphi(u_\lambda) \le \liminf_{\lambda\searrow 0} \int_\Omega \psi_\lambda(u_\lambda(t)). \tag{3.39}$$

At this point, taking the lower limit in (3.38), thanks to (3.33), (3.39), and to the lower semicontinuity of the norms, we get (3.37). The positivity of the integral at the right hand side of (3.37) is an immediate consequence of the positivity of ψ_λ. □

Remark I.3.1. It can be proved that in fact in formula (3.37) the equality holds. However throughout the following chapter we will use only the weaker form with the inequality sign.

In the limit procedure of the proof of Theorem I.3.1, we have used the strong convergence of the sequence j_λ in $L^2(Q)$. However in most cases strong convergences are very hard to obtain, therefore we need some criterion which helps us to pass to the limit when we have only weak convergences.

Lemma I.3.1. *Let $Q := \Omega\times]0,T[$, where $\Omega \subseteq \mathbb{R}^d$ and $T > 0$. Let α be a maximal monotone operator in $\mathbb{R}$. Let $(u_\lambda)_{\lambda>0}$ and $(v_\lambda)_{\lambda>0}$ be two families in $L^2(Q)$ and $u, v \in L^2(Q)$ such that $u_\lambda \rightharpoonup u$ and $v_\lambda \rightharpoonup v$ in $L^2(Q)$ and $v_\lambda \in \alpha(u_\lambda)$ a.e. in Q for all $\lambda > 0$. Then the following propositions hold.*

(i) $\liminf_{\lambda\searrow 0}(u_\lambda, v_\lambda)_{L^2(Q)} \le (u, v)_{L^2(Q)} \quad \Longrightarrow \quad v \in \alpha(u)$ *a.e. in* Q

(ii) $\limsup_{\lambda\searrow 0}(u_\lambda, v_\lambda)_{L^2(Q)} \le (u, v)_{L^2(Q)} \quad \Longrightarrow \quad \lim_{\lambda\searrow 0}(u_\lambda, v_\lambda)_{L^2(Q)} = (u, v)_{L^2(Q)}$

Proof.

(i)

As $v_\lambda \in \alpha(u_\lambda)$ a.e. in Q for all $\lambda > 0$, we have that $(v_\lambda(x,t)-s)(u_\lambda(x,t)-r) \ge 0$ for almost all $(x,t) \in Q$ and for all $r \in D(\alpha)$, $s \in \alpha(r)$. Therefore we infer that

$$\int_Q \Big(v_\lambda(x,t) - z(x,t)\Big)\Big(u_\lambda(x,t) - w(x,t)\Big)dxdt \ge 0$$
$$\forall w, z \in L^2(Q),\ w \in D(\alpha),\ z \in \alpha(v) \text{ a.e. in } Q.$$

Now taking the lower limit in the previous inequality as $\lambda \searrow 0$, and using the hypotesis on $\liminf_\lambda (u_\lambda, v_\lambda)_{L^2(Q)}$, we find

$$\int_Q \Big(v(x,t) - z(x,t)\Big)\Big(u(x,t) - w(x,t)\Big)dxdt \geq 0$$
$$\forall w, z \in L^2(Q),\ w \in D(\alpha),\ z \in \alpha(v) \text{ a.e. in } Q. \tag{3.40}$$

Now, arguing as in the last part of the proof of Theorem I.3.1, we deduce that $(v(x,t) - s)(u(x,t) - r) \geq 0$ for almost all $(x,t) \in Q$ and for all $r \in \mathbb{R}$, $s \in \alpha(r)$, and therefore we get $v \in \alpha(u)$ a.e. in Q.

(ii)

From (i) we have that $v \in \alpha(u)$ a.e. in Q, hence $(v - v_\lambda)(u - u_\lambda) \geq 0$ a.e. in Q and for all λ. We integrate over Q this inequality and we get that

$$\int_Q (v(x,t) - v_\lambda(x,t))(u(x,t) - u_\lambda(x,t))dxdt \geq 0,$$

hence taking the upper limit as $\lambda \searrow 0$, we find that $\liminf_{\lambda \searrow 0}(u_\lambda, v_\lambda)_{L^2(Q)} \geq (u, v)_{L^2(Q)}$, and (ii) follows. □

I.4 Approximation

In this section we study the perturbation of the Stefan problem given by (3.13)–(3.14). As we recalled in the previous section this perturbed problem is often called relaxed Stefan problem and its analysis will be also considered in Chapter II. We replace the sign graph by a more general maximal monotone operator γ. To be more precise we consider

$$\gamma : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R}) \quad \text{maximal monotone}, \qquad \beta := \alpha^{-1}, \tag{4.1}$$

and we define $\varphi : \mathbb{R} \longrightarrow \widetilde{\mathbb{R}}$ by setting

$$\varphi(r) := \begin{cases} \displaystyle\int_{r_0}^{r} \beta_0(s)ds & \text{if } r \in \overline{D(\beta)} \\ \infty & \text{if } r \notin \overline{D(\beta)} \end{cases}. \tag{4.2}$$

Throughout this section we will use the notations defined by (2.1)–(2.11). The data of the problem are the following.

$$f \in L^2(0,T;V') + L^1(0,T;H), \tag{4.3}$$
$$\theta_D \in H^1(0,T;H), \tag{4.4}$$
$$\theta_0 \in H, \tag{4.5}$$
$$\chi_0 \in H, \quad \varphi(\chi_0) \in L^1(\Omega). \tag{4.6}$$

Now we can give the precise weak formulation of the perturbed Stefan problem.

Problem $(\mathbf{PS}_\varepsilon)$. Let $\varepsilon > 0$ and assume that (2.1)–(2.11) and (4.1)–(4.6) hold. Find a pair $(\theta_\varepsilon, \chi_\varepsilon)$ such that

$$\theta_\varepsilon \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{4.7}$$
$$\theta_\varepsilon' \in L^2(0,T;V') + L^1(0,T;H) \tag{4.8}$$
$$\chi_\varepsilon \in H^1(0,T;H), \tag{4.9}$$
$$\exists \xi_\varepsilon \in L^2(Q), \qquad \xi_\varepsilon \in \beta(\chi_\varepsilon) \qquad \text{a.e. in } Q, \tag{4.10}$$
$$(\theta_\varepsilon + \chi_\varepsilon)' + A\theta_\varepsilon = f \qquad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{4.11}$$
$$\varepsilon\chi_\varepsilon' + \xi_\varepsilon = \theta_\varepsilon + \theta_D \qquad \text{a.e. in } Q, \tag{4.12}$$
$$\theta_\varepsilon(0) = \theta_0, \quad \chi_\varepsilon(0) = \chi_0 \qquad \text{a.e. in } \Omega. \tag{4.13}$$

♠

Remark I.4.1. Notice that the two conditions in (4.13) make sense because, thanks to (4.7)–(4.9), θ_ε and χ_ε belong to $C([0,T];H)$.

Before starting with the analysis of Problem $(\mathbf{PS}_\varepsilon)$, we introduce a notation which we will use throughout the subsequent chapters. Given Banach space E and a map $v :]0,T[\longrightarrow E$, we define the function $I_0 v : [0,T] \longrightarrow E$ by

$$(I_0 v)(t) := \int_0^t v(s)ds, \qquad t \in [0,T]. \tag{4.14}$$

The following property is obvious, but we explicitly state it in a lemma in order to avoid repetition in the sequel.

Lemma I.4.1. *If E is a normed space and $v \in L^2(0,T;E)$, then we have*

$$\|(I_0 v)(t)\|_E \le t^{1/2} \|v\|_{L^2(0,t;E)} \quad \forall t \in [0,T]. \tag{4.15}$$

Proof. Exploiting the Hölder inequality we find that for all $t \in [0,T]$

$$\begin{aligned} \|(I_0 v)(t)\|_E^2 &= \left\| \int_0^t v(s)ds \right\|_E^2 \le \left(\int_0^t \|v(s)\|_E ds \right)^2 \\ &\le \left(\|1\|_{L^2(0,t;E)} \|v\|_{L^2(0,t;E)} \right)^2 = t\|v\|_{L^2(0,t;E)}^2. \end{aligned}$$

□

Now we begin the study of the perturbed Stefan problem. We are going to solve it by means of the Banach fixed point Theorem.

Lemma I.4.2. *Let* $\varepsilon > 0$ *and assume that* (2.1)–(2.11) *and* (4.1)–(4.6) *hold. If* $\Theta_\varepsilon \in L^2(0,T;H)$*, then there exists a unique pair* $(\theta_\varepsilon, \chi_\varepsilon)$ *such that*

$$\theta_\varepsilon \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{4.16}$$

$$\theta'_\varepsilon \in L^2(0,T;V') \cap L^1(0,T;H), \tag{4.17}$$

$$\chi_\varepsilon \in H^1(0,T;H), \tag{4.18}$$

$$\exists \xi_\varepsilon \in L^2(Q), \quad \xi_\varepsilon \in \beta(\chi_\varepsilon) \quad \textit{a.e. in } Q, \tag{4.19}$$

$$\theta'_\varepsilon + A\theta_\varepsilon = f - \chi'_\varepsilon \quad \textit{in } V', \quad \textit{a.e. in }]0,T[, \tag{4.20}$$

$$\varepsilon\chi'_\varepsilon + \xi_\varepsilon = \Theta_\varepsilon + \theta_D \quad \textit{a.e. in } Q, \tag{4.21}$$

$$\theta_\varepsilon(0) = \theta_0, \quad \chi_\varepsilon(0) = \chi_0 \quad \textit{a.e. in } \Omega. \tag{4.22}$$

Proof. By Theorem I.3.1 there exists a unique function χ_ε satisfying (4.18)–(4.19), (4.21), and the initial condition in (4.22). Hence we plug the found χ_ε into (4.20) and by applying Theorem I.2.1 we infer that there is a unique θ_ε satisfying (4.16)–(4.17), (4.20), and the first condition of (4.22). □

Thanks to Lemma (I.4.2) we can define a nonlinear map $\Sigma : L^2(0,T;H) \longrightarrow L^2(0,T;H)$ which assigns to Θ_ε that function θ_ε such that the pair $(\theta_\varepsilon, \chi_\varepsilon)$ is the unique solution of (4.16)–(4.22). The next lemma yields a contracting estimate for Σ.

Lemma I.4.3. *There exists a constant* $C > 0$*, depending only on* T *and* ε*, such that*

$$\|\Sigma(\Theta_1) - \Sigma(\Theta_2)\|^2_{L^2(0,t;H)} \le C \int_0^t \|\Theta_1 - \Theta_2\|^2_{L^2(0,s;H)} ds \qquad \forall t \in [0,T] \tag{4.23}$$

for any $\Theta_1, \Theta_2 \in L^2(0,T;H)$.

Proof. In order to simplify the notations we omit the subscript ε. Let (θ_i, χ_i), $i = 1,2$, satisfying (4.16)–(4.22) with Θ_ε replaced by Θ_i, $i = 1,2$. Let us fix $t \in [0,T]$. We first multiply the difference of equations (4.21) by $\chi_1 - \chi_2$ and integrate over $\Omega\times]0,s[$, with $s \in]0,t[$. Thanks to the monotonicity of β we get

$$\frac{\varepsilon}{2}\|\chi_1(s) - \chi_2(s)\|^2_H \le \delta \int_0^s \|\chi_1(\tau) - \chi_2(\tau)\|^2_H d\tau + \frac{1}{4\delta}\|\Theta_1 - \Theta_2\|^2_{L^2(0,s;H)},$$

where $\delta > 0$ will be fixed later. Now integrating the previous inequality from 0 to t we deduce that

$$\begin{aligned}
&\frac{\varepsilon}{2}\|\chi_1 - \chi_2\|^2_{L^2(0,t;H)} \\
&\le \delta \int_0^t \int_0^s \|\chi_1(\tau) - \chi_2(\tau)\|^2_H d\tau ds + \frac{1}{4\delta}\int_0^t \|\Theta_1 - \Theta_2\|^2_{L^2(0,s;H)} ds \\
&\le \delta \int_0^t \|\chi_1 - \chi_2\|^2_{L^2(0,t;H)} ds + \frac{1}{4\delta}\int_0^t \|\Theta_1 - \Theta_2\|^2_{L^2(0,s;H)} ds \\
&\le \delta T \|\chi_1 - \chi_2\|^2_{L^2(0,t;H)} + \frac{1}{4\delta}\int_0^t \|\Theta_1 - \Theta_2\|^2_{L^2(0,s;H)} ds,
\end{aligned}$$

thus choosing $\delta = \varepsilon/(4T)$ we find

$$\|\chi_1 - \chi_2\|^2_{L^2(0,t;H)} \le \frac{4T}{\varepsilon^2}\int_0^t \|\Theta_1 - \Theta_2\|^2_{L^2(0,s;H)} ds. \tag{4.24}$$

On the other hand Proposition I.2.3 yields that

$$\begin{aligned}
\|\theta_1 - \theta_2\|^2_{L^2(0,t;H)} \le C\|\chi_1' - \chi_2'\|^2_{L^2(0,t;H)} &= C\int_0^t \|(I_0(\chi_1 - \chi_2))(s)\|^2_H ds \\
&\le C\int_0^t s\|(\chi_1 - \chi_2)(s)\|^2_H ds \le Ct\|\chi_1 - \chi_2\|^2_{L^2(0,t;H)}, \qquad (4.25)
\end{aligned}$$

for some positive C depending only on T. Therefore combining (4.24) and (4.25) we find (4.23). □

Theorem I.4.1. *Problem* $\mathbf{(PS_\varepsilon)}$ *has a unique solution.*

Proof. First of all let us observe that a couple $(\theta_\varepsilon, \chi_\varepsilon)$ is a solution to Problem $\mathbf{(PS_\varepsilon)}$ if and only if θ_ε is a fixed point of the map Σ. In fact on one hand if $(\theta_\varepsilon, \chi_\varepsilon)$ solves Problem $\mathbf{(PS_\varepsilon)}$ then it is the unique pair satisfying (4.16)–(4.22) with Θ replaced by θ_ε, and therefore θ_ε is a fixed point of Σ. On the other hand it is clear that $(\theta_\varepsilon, \chi_\varepsilon)$ solves Problem $\mathbf{(PS_\varepsilon)}$ whenever θ_ε is a fixed point of Σ. Now let $\Theta_1, \Theta_2 \in L^2(0,T;H)$. Then thanks to Lemma I.4.3 we have that

$$\|\Sigma(X_1) - \Sigma(X_2)\|^2_{L^2(0,t;H)} \le Ct\|X_1 - X_2\|^2_{L^2(0,t;H)} \qquad \forall t \in [0,T]. \tag{4.26}$$

Now we apply again Lemma I.4.3 with Θ_i replaced by $\Sigma(\Theta_i)$, $i = 1, 2$, and use (4.26) to obtain

$$\begin{aligned}
\|\Sigma^2(\Theta_1) - \Sigma^2(\Theta_2)\|^2_{L^2(0,t;H)} &\le C\int_0^t \|\Sigma(\Theta_1) - \Sigma(\Theta_2)\|^2_{L^2(0,s;H)} ds \\
&\le \frac{C^2t^2}{2}\|\Theta_1 - \Theta_2\|^2_{L^2(0,t;H)} \qquad \forall t \in [0,T].
\end{aligned}$$

Thus by induction it is easy to prove that

$$\|\Sigma^n(\Theta_1) - \Sigma^n(\Theta_2)\|^2_{L^2(0,T;H)} \leq \frac{C^nT^n}{n!}\|\Theta_1 - \Theta_2\|^2_{L^2(0,T;H)}$$

for all $n \in \mathbb{N}_*$. This proves that for n sufficiently large, Σ^n is a (strict) contraction in the complete metric space $L^2(0,T;H)$ and the theorem is proved. $\square$

Now we conclude this section stating and proving two simple lemmas which will be repeteadly used in the sequel.

Lemma I.4.4. *Let $\phi_0 \in H$ and $v \in L^2(0,T;V)$. Then for all $t \in (0,T)$ and $\delta > 0$ we have*

$$\int_0^t \langle \phi_0, v(s)\rangle ds \leq \delta\|v\|^2_{L^2(0,t;H)} + \frac{t}{4\delta}\|\phi_0\|^2_H. \tag{4.27}$$

Proof. Since $\phi_0 \in H$ we can write

$$\begin{aligned}\int_0^t \langle \phi_0, v(s)\rangle ds &= \int_0^t (\phi_0, v(s))_H ds \leq \int_0^t \|\phi_0\|_H\|v(s)\|_H ds\\ &\leq \delta\int_0^t \|v(s)\|^2_H ds + \frac{1}{4\delta}\int_0^t \|\phi_0\|^2_H ds \leq \delta\|v\|^2_{L^2(0,t;H)} + \frac{t}{4\delta}\|\phi_0\|^2_H.\end{aligned}$$

$\square$

Lemma I.4.5. *Let $\phi \in L^2(0,T;V') + L^1(0,T;H)$ and let $\phi_{V'} \in L^2(0,T;V')$ and $\phi_H \in L^1(0,T;H)$ such that $\phi = \phi_{V'} + \phi_H$. Let $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Then for all $t \in (0,T)$ and $\delta > 0$ we have*

$$\begin{aligned}\int_0^t \big\langle \phi(s), v(s)\big\rangle ds \leq \delta\|v\|^2_{L^2(0,t;H)} + \delta\|\nabla v\|^2_{L^2(0,t;\mathbf{H})} + \frac{1}{4\delta}\|\phi_{V'}\|^2_{L^2(0,t:V')}\\ + \int_0^t \|\phi_H(s)\|_H\|v(s)\|_H ds\end{aligned}$$

and

$$\begin{aligned}\int_0^t \big\langle (I_0\phi)(s), v(s)\big\rangle ds \leq \delta(1+t+t^2/2)\|v\|^2_{L^2(0,t;H)} + \delta\|\nabla(I_0v)(t)\|^2_{\mathbf{H}}\\ + \delta\|\nabla(I_0v)\|^2_{L^2(0,t;\mathbf{H})} + \frac{\min\{1,t\}}{4\delta}\|\phi\|^2_{L^2(0,t:V')+L^1(0,t;H)}.\end{aligned}$$

Proof. The proof of the first inequality was already done in Proposition I.2.2, formula (2.27). Now we prove the second estimate. We have

$$\int_0^t \big\langle (I_0\phi)(s), v(s)\big\rangle ds = \int_0^t \Big(\big\langle (I_0\phi_{V'})(s), v(s)\big\rangle + ((I_0\phi_H)(s), v(s))\Big)\, ds. \tag{4.28}$$

Let us observe that by means of an integration by parts, and using also Lemma I.4.1 we have

$$\begin{aligned}
&\int_0^t \big\langle (I_0\phi_{V'})(s), v(s)\big\rangle ds \\
&= \big\langle (I_0\phi_{V'})(t), (I_0 v)(t)\big\rangle - \int_0^t \big\langle \phi_{V'}(s), (I_0 v)(s)\big\rangle ds \\
&\le \|(I_0\phi)(t)\|_{V'}\|(I_0 v)(s)\|_V + \int_0^t \|\phi_{V'}(s)\|_{V'}\|(I_0 v)(s)\|_V ds \\
&= \|(I_0\phi)(t)\|_{V'}\left(\|(I_0 v)(t)\|_H^2 + \|\nabla (I_0 v)(t)\|_{\mathbf{H}}^2\right)^{1/2} \\
&\quad + \int_0^t \|\phi_{V'}(s)\|_{V'}\left(\|(I_0 v)(s)\|_H^2 + \|\nabla (I_0 v)(s)\|_{\mathbf{H}}^2\right)^{1/2} ds \\
&\le \delta\left(t\|v\|_{L^2(0,t;H)}^2 + \|\nabla (I_0 v)(t)\|_{\mathbf{H}}^2\right) + \frac{1}{4\delta}\|(I_0\phi_{V'})(t)\|_{V'}^2 \\
&\quad + \delta\int_0^t \left(s\|v\|_{L^2(0,s;H)}^2 + \|\nabla (I_0 v)(s)\|_{\mathbf{H}}^2\right) ds + \frac{1}{4\delta}\|\phi_{V'}\|_{L^2(0,t;V')}^2 \\
&\le \delta\left(t\|v\|_{L^2(0,t;H)}^2 + \|\nabla (I_0 v)(t)\|_{\mathbf{H}}^2\right) + \frac{t}{4\delta}\|\phi_{V'}\|_{L^2(0,t;V')}^2 + \\
&\quad + \delta\int_0^t \left(s\|v\|_{L^2(0,t;H)}^2 + \|\nabla (I_0 v)(s)\|_{\mathbf{H}}^2\right) ds + \frac{1}{4\delta}\|\phi_{V'}\|_{L^2(0,t;V')}^2 .
\end{aligned} \tag{4.29}$$

On the other hand

$$\begin{aligned}
&\int_0^t ((I_0\phi_H)(s), v(s)) ds \le \int_0^t \|(I_0\phi_H)(s)\|_H \|v(s)\|_H ds \\
&\le \delta\|v\|_{L^2(0,t;H)}^2 + \frac{1}{4\delta}\|I_0\phi_H\|_{L^2(0,t;H)}^2 \le \delta\|v\|_{L^2(0,t;H)}^2 + \frac{t}{4\delta}\|\phi_H\|_{L^1(0,t;H)}^2 .
\end{aligned} \tag{4.30}$$

Hence collecting (4.28)–(4.30) we achieve the statement. □

I.5 Existence and uniqueness

In this section we perform some a priori estimates on the solution $(\theta_\varepsilon, \chi_\varepsilon)$ of Problem $(\mathbf{PS}_\varepsilon)$, and we recover the solution (θ, χ) of the Stefan problem as the limit of $(\theta_\varepsilon, \chi_\varepsilon)$ as ε goes to zero. We will essentially follow the procedure outlined in Section I.3. Of course we set all the data in the framework of the previous section. The precise variational formulation of the Stefan problem reads as follows.

Problem (S). Assume that (2.1)–(2.11) and (4.1)–(4.6) hold. Find a pair (θ, χ) such that

$$\theta \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{5.1}$$

$$(\theta+\chi)' \in L^2(0,T;V') + L^1(0,T;H) \tag{5.2}$$

$$\chi \in L^2(Q), \tag{5.3}$$

$$(\theta+\chi)' + A\theta = f \qquad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{5.4}$$

$$\chi \in \gamma(\theta + \theta_D) \qquad \text{a.e. in } Q, \tag{5.5}$$

$$(\theta+\chi)(0) = \theta_0 + \chi_0 \qquad \text{in } V'. \tag{5.6}$$

♠

Lemma I.5.1. *Problem* **(S)** *has at most one solution.*

Proof. Let (θ_1, χ_1) and (θ_2, χ_2) be two solutions. Let us set $\widetilde{\theta} := \theta_1 - \theta_2$ $\widetilde{\chi} := \chi_1 - \chi_2$ and let $t \in [0,T]$. First of all let us subtract the respective equations (5.4) for (θ_i, χ_i), $i = 1,2$, from each other and integrate the difference from 0 to s, where $s \in [0,t]$. Then we test the result by $\widetilde{\theta}(s)$ and integrate again over $[0,t]$. Finally we find

$$\|\widetilde{\theta}\|^2_{L^2(0,t;H)} + \int_0^t \int_\Omega \widetilde{\chi}(s)\widetilde{\theta}(s) ds + \frac{1}{2}\|\nabla(I_0\widetilde{\theta})(t)\|^2_{\mathbf{H}} = 0.$$

From inclusion (5.5) and from the monotonicity of γ we infer that $\widetilde{\chi}\widetilde{\theta} = (\chi_1 - \chi_2)(\theta_1 + \theta_D - \theta_2 - \theta_D) \geq 0$ a.e. in Q, hence

$$\int_0^t \int_\Omega \widetilde{\chi}(s)\widetilde{\theta}(s) ds \geq 0. \tag{5.7}$$

Therefore we infer that $\|\widetilde{\theta}\|_{L^2(0,T;H)} = 0$, i.e. $\theta_1 = \theta_2$ a.e. in Q. Then by a comparison follows that $\chi_1 = \chi_2$ a.e. in Q. □

To get an existence theorem for the Stefan problem we need some further assumption on the multivalued map γ. To be precise we need that

$$\exists C_\gamma > 0 \quad : \quad |s| \leq C_\gamma(|r| + 1) \quad \forall r \in D(\gamma), \ \forall s \in \gamma(r). \tag{5.8}$$

Notice that under assumptions (5.8) we have that $D(\gamma) = \mathbb{R}$. Now we look for the a priori estimates needed to pass to the limit in the Problem $(\mathbf{PS}_\varepsilon)$ and to solve the Stefan Problem $(\mathbf{S})$.

Proposition I.5.1. *Assume* (5.8)*. Let* $(\theta_\varepsilon, \chi_\varepsilon)$ *be the solution to Problem* $(\mathbf{PS}_\varepsilon)$ *and let* ξ_ε *satisfying* (4.10) *and* (4.12)*. Then there exists a constant* $C > 0$*, independent of* ε*, such that*

$$\begin{aligned} &\varepsilon^{1/2}\|\chi_\varepsilon'\|_{L^2(0,T;H)} + \|\theta_\varepsilon\|_{L^\infty(0,T;H)} + \|\nabla\theta_\varepsilon\|_{L^2(0,T;\mathbf{H})} \\ &\quad + \|\chi_\varepsilon\|_{L^2(0,T;H)} + \|\xi_\varepsilon\|_{L^2(0,T;H)} \le C. \end{aligned} \tag{5.9}$$

Proof.

1st estimate

The first estimate we are going to recover is obtained testing the equation (4.11) by θ_ε. We get that

$$\begin{aligned} &\frac{1}{2}\|\theta_\varepsilon(t)\|_H^2 + \int_0^t\int_\Omega \chi_\varepsilon'(s)\theta_\varepsilon(s)ds + \|\nabla\theta_\varepsilon\|_{L^2(0,t;\mathbf{H})}^2 \\ &= \frac{1}{2}\|\theta_0\|_H^2 + \int_0^t \langle f(s), \theta_\varepsilon(s)\rangle ds \end{aligned} \tag{5.10}$$

Now let $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^1(0,T;H)$ such that $f = f_{V'} + f_H$. Using Hölder and Young inequalities to treat the right hand side of the last equation, we infer that there exists a constant $C > 0$, independent of ε, but depending only on T and on the data f, θ_D, θ_0, such that

$$\begin{aligned} &\frac{1}{2}\|\theta_\varepsilon(t)\|_H^2 + \int_0^t\int_\Omega \chi_\varepsilon'(s)\theta_\varepsilon(s)ds + \frac{1}{2}\|\nabla\theta_\varepsilon\|_{L^2(0,t;\mathbf{H})}^2 \\ &\le C + C\int_0^t \|\theta_\varepsilon(s)\|_H^2 ds + \int_0^t \|f_H(s)\|_H\|\theta_\varepsilon(s)\|_H ds. \end{aligned} \tag{5.11}$$

2nd estimate

Let us choose r_0 such that $0 \in \beta(r_0)$ (hence $\beta_0(r_0) = 0$, recall that $D(\gamma) = \mathbb{R}$). From Corollary I.3.1 we deduce that

$$\varepsilon\|\chi_\varepsilon'\|_{L^2(0,t;H)}^2 + \int_\Omega (\varphi\circ\chi_\varepsilon)(t) \le \int_\Omega \varphi\circ\chi_0 + \int_0^t\int_\Omega \theta_\varepsilon(s)\chi_\varepsilon'(s)ds, \tag{5.12}$$

but for $r \in D(\beta)$ we have

$$\begin{aligned}\varphi(r) &= \int_{r_0}^{r} \beta_0(s)ds \geq \frac{1}{C_\gamma}\int_{r_0}^{r}(|s| - C_\gamma)ds \geq \frac{1}{C_\gamma}\int_{r_0}^{r}(s - C_\gamma)ds \\ &= \frac{1}{2C_\gamma}\big((r - C_\gamma)^2 - (r_0 - C_\gamma)^2\big),\end{aligned}$$

therefore there exists a constant $C'_\gamma > 0$ sufficiently large, such that

$$\varphi(r) \geq \frac{1}{C'_\gamma} r^2 - C'_\gamma \qquad \forall r \in D(\beta). \tag{5.13}$$

Hence from (5.12), using (5.13) and thanks to an integration by parts, we get

$$\begin{aligned}&\varepsilon\|\chi'_\varepsilon\|^2_{L^2(0,t;H)} + C'_\gamma\|\chi_\varepsilon(t)\|^2_H \\ &\leq C + \int_0^t\int_\Omega \theta_\varepsilon(s)\chi'_\varepsilon(s)ds + \int_\Omega \theta_D(t)\chi_\varepsilon(t) - \int_\Omega \theta_D(0)\chi_0 - \int_0^t\int_\Omega \theta'_D(s)\chi_\varepsilon(s)ds.\end{aligned}$$

Next using Hölder and Young inequalities in the previous formula we obtain

$$\begin{aligned}&\varepsilon\|\chi'_\varepsilon\|^2_{L^2(0,t;H)} + \frac{C'_\gamma}{2}\|\chi_\varepsilon(t)\|^2_H \\ &\leq C\left(1 + \int_0^t (\|\theta_\varepsilon(t)\|^2_H + \|\chi_\varepsilon(t)\|^2_H)\right) + \int_0^t\int_\Omega \theta_\varepsilon(s)\chi'_\varepsilon(s)ds. \end{aligned}\tag{5.14}$$

Adding the last inequality and (5.11), we observe that there is a cancellation and, applying the generalized Gronwall Lemma, we infer that

$$\varepsilon\|\chi'_\varepsilon\|^2_{L^2(0,t;H)} + \|\chi_\varepsilon\|^2_{L^\infty(0,t;H)} + \|\theta_\varepsilon\|^2_{L^\infty(0,t;H)} + \|\nabla\theta_\varepsilon\|^2_{L^2(0,t;\mathbf{H})} \leq C.$$

3^{rd} estimate

By a comparison in the equation (4.12) we get

$$\begin{aligned}\|\xi_\varepsilon\|_{L^2(0,t;H)} &= \|\theta_\varepsilon + u - \varepsilon\chi'_\varepsilon\|_{L^2(0,t;H)} \\ &\leq \|\theta_\varepsilon\|_{L^2(0,t;H)} + \|u\|_{L^2(0,t;H)} + \varepsilon^{1/2}\|\varepsilon^{1/2}\chi'_\varepsilon\|_{L^2(0,t;H)} \leq C.\end{aligned}\tag{5.15}$$

□

Now we can finally prove the main theorem of Chapter I

Theorem I.5.1. *Assume* (5.8). *Then Problem* **(S)** *has a unique solution.*

Proof. From Proposition I.5.1 we deduce that there exist three function θ, χ, and ξ satisfying (5.1)–(5.3) and such that, at least for subsequences, the following convergences hold.

$$\theta_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^2(0,T;V) \cap L^\infty(0,T;H), \tag{5.16}$$

$$A\theta_\varepsilon \rightharpoonup A\theta \quad \text{in } L^2(0,T;V'), \tag{5.17}$$

$$\varepsilon\chi_\varepsilon' \to 0 \quad \text{in } L^2(0,T;H), \tag{5.18}$$

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{in } L^2(0,T;H), \tag{5.19}$$

$$\chi_\varepsilon \rightharpoonup \chi \quad \text{in } L^2(0,T;H). \tag{5.20}$$

Now let $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^2(0,T;H)$ such that $f = f_{V'} + f_H$. Then it follows from estimate (5.9) that $(\theta_\varepsilon + \chi_\varepsilon)' - f_H$ is bounded in $L^2(0,T;V')$. Then, thanks also to Proposition A.2.4, it is easy to see that

$$(\theta_\varepsilon + \chi_\varepsilon)' - f_H \rightharpoonup (\theta + \chi)' - f_H \quad \text{in } L^2(0,T;V').$$

Observe also that convergence (5.16) implies that $I_0\theta_\varepsilon \rightharpoonup I_0\theta$ in $H^1(0,T;V)$, from which it follows that

$$(I_0\theta_\varepsilon)(t) \rightharpoonup (I_0\theta)(t) \quad \text{in } V, \quad \forall t \in [0,T].$$

Therefore taking the limit in equation (4.11) we find that (5.4) and (5.6) are satisfied and that

$$\xi = \theta + \theta_D \quad \text{a.e. in } Q. \tag{5.21}$$

The proof is complete if we show that the nonlinear relation (5.5) is valid, i.e. if $\xi \in \beta(\chi)$ a.e. in Q. Thanks to Lemma I.3.1 it is enough to show that

$$\limsup_{\varepsilon \searrow 0} \int_Q \xi_\varepsilon \chi_\varepsilon \le \int_Q \xi\chi. \tag{5.22}$$

In order to prove (5.22), we use (4.12) and write

$$\int_Q \xi_\varepsilon \chi_\varepsilon = \int_Q (\theta_\varepsilon + u - \varepsilon\chi_\varepsilon')\chi_\varepsilon. \tag{5.23}$$

Now notice that

$$\begin{aligned} \int_Q \chi_\varepsilon\theta_\varepsilon &= \int_0^T \big\langle \theta_\varepsilon(s), \chi_\varepsilon(s) \big\rangle ds \\ &= \int_0^T \big\langle \theta_0 + \chi_0 + (I_0 f)(s) - \theta_\varepsilon(s) - A(I_0\theta_\varepsilon)(s), \theta_\varepsilon(s) \big\rangle ds \\ &= \int_0^T \big\langle \theta_0 + \chi_0 + (I_0 f)(s), \theta_\varepsilon(s) \big\rangle ds - \|\theta_\varepsilon\|^2_{L^2(0,T;H)} - \frac{1}{2}\|\nabla(I_0\theta_\varepsilon)(T)\|^2_{\mathbf{H}}, \end{aligned} \tag{5.24}$$

therefore by the lower semicontinuity of the norms

$$
\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\int_Q \chi_\varepsilon\theta_\varepsilon \\
&\le \int_0^T \big\langle \theta_0+\chi_0+(I_0 f)(s),\theta(s)\big\rangle ds - \|\theta\|^2_{L^2(0,T;H)} - \frac{1}{2}\|\nabla(I_0\theta)(T)\|^2_{\mathbf{H}} \\
&= \int_0^T \big\langle \theta_0+\chi_0+(I_0 f)(s)-\theta(s)-A(I_0\theta)(s),\theta(s)\big\rangle ds = \int_0^T \chi\theta \qquad (5.25)
\end{aligned}
$$

And consequently by the convergences established above

$$
\begin{aligned}
\limsup_{\varepsilon\searrow 0}\int_Q \xi_\varepsilon\chi_\varepsilon &\le \limsup_{\varepsilon\searrow 0}\int_Q \theta_\varepsilon\chi_\varepsilon + \limsup_{\varepsilon\searrow 0}\int_Q \theta_D\chi_\varepsilon - \liminf_{\varepsilon\searrow 0}\int_Q (\varepsilon\chi_\varepsilon')\chi_\varepsilon \\
&\le \int_Q \theta\chi + \int_Q \theta_D\chi = \int_Q (\theta+\theta_D)\chi = \int_Q \xi\chi, \qquad (5.26)
\end{aligned}
$$

and the theorem is proved. □

Chapter II

Phase relaxation

In this chapter, on a basis of physical motivations, we introduce two generalizations of the Stefan model of phase transitions. The first one is a classical model introduced by Visintin and the second is a more recent model, which has been proposed by Visintin in [38] and has been studied by the author in [30]. Both models aim to represent the physical situation where undercooling or superheating effects occur. Indeed the formulation of the Stefan model presented in Chapter I does not take into account these phenomena.

In the first section we review some well known results on the first model, exploiting the theorems proved in the first chapter. In Section 2 we provide a physical justification of the second model and prove an existence and uniqueness result. In Section 3 we study the asymptotic behaviour of the solution of the relaxed problem when the relaxation parameter goes to zero.

II.1 Kinetic undercooling and phase relaxation

The heat equation is a relation that describes nonequilibrium, but the constitutive relation (1.18) of Section I.1 between the phase and the temperature is an equilibrium condition. Indeed it is based on the assumption that the moving interface between the two phases is at local thermodynamical equilibrium.

Instead it seems more reasonable to assume that the phase transition is driven by a nonequilibrium condition, in such a way that *dynamic superheating* or *undercooling effects* can be taken into account.

Quoting Chalmers [9] we see that "if the interface is not at the equilibrium temperature, then either melting or solidification occurs, at a rate which increases with the difference between the actual temperature and the equilibrium temperature. For small departures from equilibrium the rate is approximatively proportional to the departure". According to Visintin ([37]) the implication stated by Chalmers can be inverted: "phase transition occurs only if the solid-liquid interface is not at the equilibrium temperature. So phase transition is triggered by deviation from the equilibrium temperature, and exchange of latent heat at the interface is the effect of phase transition".

Therefore it is reasonable to replace the Stefan condition by the relaxation dynamics (cf. Visintin [36])

$$\varepsilon\frac{\partial\chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q = \Omega\times]0,T[, \tag{1.1}$$

ε being a small relaxation parameter. In this way we are led to study the following generalization of the Stefan problem, that is usually called *relaxed Stefan problem* or *Stefan problem with phase relaxation*:

$$\frac{\partial(\theta+\chi)}{\partial t} - \Delta\theta = f \qquad \text{in } Q, \tag{1.2}$$

$$\varepsilon\frac{\partial\chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q, \tag{1.3}$$

complemented by suitable initial and boundary conditions.

We already analyzed this problem in the previous chapter, to the aim of solving the Stefan problem. For the sake of clarity let us restate again the main result of Sections I.4 and I.5. For convenience we also recall the functional framework where we set the problem.

$$d \in \mathbb{N}_*, \quad \Omega \text{ is a bounded open and connected subset of } \mathbb{R}^d, \tag{1.4}$$

$$\Gamma := \partial\Omega \text{ is of Lipschitz class}, \tag{1.5}$$

$$\mathbf{n} \text{ is the outward normal unit vector to } \Omega, \tag{1.6}$$

$$\Gamma_D \text{ and } \Gamma_N \text{ are open subsets of } \Gamma, \tag{1.7}$$

$$\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma, \quad \Gamma_D \cap \Gamma_N = \varnothing, \quad \overline{\Gamma}_D \cap \overline{\Gamma}_N \text{ is of Lipschitz class}, \tag{1.8}$$

$$Q := \Omega\times]0,T[, \text{ where } T \in]0,\infty[. \tag{1.9}$$

Then we set

$$H := L^2(\Omega), \qquad V := H^1_{\Gamma_D}(\Omega), \tag{1.10}$$

endowed with their usual inner product. The inner product in H will be denoted by $(\cdot,\cdot)_H$, whereas ${}_{V'}\langle\cdot,\cdot\rangle_V$ stands for the duality pairing between V' and V. Moreover we set

$$\mathbf{H} := L^2(\Omega;\mathbb{R}^d), \tag{1.11}$$

Concerning the assumptions on the data, we have

$$\gamma : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R}) \quad \text{maximal monotone}, \qquad \beta := \alpha^{-1}, \tag{1.12}$$

and we define $\varphi : \mathbb{R} \longrightarrow \widetilde{\mathbb{R}}$ by setting

$$\varphi(r) := \begin{cases} \displaystyle\int_{r_0}^{r} \beta_0(s)ds & \text{if } r \in \overline{D(\beta)} \\ \infty & \text{if } r \notin \overline{D(\beta)} \end{cases}. \tag{1.13}$$

Finally we suppose that

$$f \in L^2(0,T;V') + L^1(0,T;H), \tag{1.14}$$
$$\theta_D \in H^1(0,T;H) \tag{1.15}$$
$$\theta_0 \in H, \tag{1.16}$$
$$\chi_0 \in H, \quad \varphi(\chi_0) \in L^1(\Omega). \tag{1.17}$$

We recall that in order to get the existence of a solution of the Stefan problem, we needed the monotone map γ to be sublinear, i.e.

$$\exists C_\gamma > 0 \qquad : \qquad |s| \leq C_\gamma(|r|+1) \quad \forall r \in D(\gamma),\ \forall s \in \gamma(r). \tag{1.18}$$

The *Stefan problem with phase relaxation* then reads as follows.

Problem (S1$_\varepsilon$). Let $\varepsilon > 0$ and assume that (1.4)–(1.18) hold. Find a pair $(\theta_\varepsilon, \chi_\varepsilon)$ such that

$$\theta_\varepsilon \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{1.19}$$
$$\theta_\varepsilon' \in L^2(0,T;V') + L^1(0,T;H) \tag{1.20}$$
$$\chi_\varepsilon \in H^1(0,T;H), \tag{1.21}$$
$$\exists \xi_\varepsilon \in L^2(Q), \qquad \xi_\varepsilon \in \beta(\chi_\varepsilon) \qquad \text{a.e. in } Q, \tag{1.22}$$
$$(\theta_\varepsilon + \chi_\varepsilon)' + A\theta_\varepsilon = f \qquad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{1.23}$$
$$\varepsilon\chi_\varepsilon' + \xi_\varepsilon = \theta_\varepsilon + \theta_D \qquad \text{a.e. in } Q, \tag{1.24}$$
$$\theta_\varepsilon(0) = \theta_0, \quad \chi_\varepsilon(0) = \chi_0 \qquad \text{a.e. in } \Omega. \tag{1.25}$$

♠

We also recall the variational formulation of the Stefan problem.

Problem (S). Assume that (1.4)–(1.18) hold. Find a pair (θ, χ) satisfying the following conditions.

$$\theta \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{1.26}$$
$$(\theta+\chi)' \in L^2(0,T;V') + L^1(0,T;H) \tag{1.27}$$
$$\chi \in L^2(Q), \tag{1.28}$$
$$(\theta+\chi)' + A\theta = f \quad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{1.29}$$
$$\chi \in \gamma(\theta+\theta_D) \quad \text{a.e. in } Q, \tag{1.30}$$
$$(\theta+\chi)(0) = \theta_0 + \chi_0 \quad \text{in } V'. \tag{1.31}$$

♠

Therefore Theorem I.4.1, Lemma I.5.1, and the proof of Theorem I.5.1 yield the following

Theorem II.1.1. *For any $\varepsilon > 0$ there exists a unique solution $(\theta_\varepsilon, \chi_\varepsilon)$ of Problem* $(\mathbf{S1}_\varepsilon)$. *Moreover there exists a pair (θ, χ) such that*

$$\theta_\varepsilon \overset{*}{\rightharpoonup} \theta \quad \text{in } L^2(0,T;V) \cap L^\infty(0,T;H), \tag{1.32}$$
$$\chi_\varepsilon \rightharpoonup \chi \quad \text{in } L^2(0,T;H). \tag{1.33}$$

and (θ, χ) is the unique solution of the Stefan Problem $(\mathbf{S})$.

In Chapter I we studied Problem $(\mathbf{S1}_\varepsilon)$ in order to get an existence result for the Stefan Problem $(\mathbf{S})$. In this section we have seen that this relaxed problem has also a physical motivation, but let us remark that also the previous convergence result is significant from a physical point of view. In fact in most of applications the relaxation parameter ε is very small with respect to the used lenght scale: the faster the process of phase transition, the smaller ε. Therefore it appears quite natural to wonder whether the solution of the relaxed problem converges, in a suitable topology, to the solution of the Stefan problem when the coefficient ε vanishes. Theorem II.1.1 gives an affermative answer to this question.

II.2 A "probabilistic" model of phase relaxation

Now we formulate an alternative model of phase relaxation for the Stefan problem which has a natural probabilistic interpretation. This model was formulated by Visintin in [38]. In this section we propose a physical motivation of this model and its rigorous analysis, following essentially a paper due the author [30]. In fact we slightly generalize the results proved in [30].

Let us recall that in Section I.3 we observed that, in order to achieve enough a priori estimates of the solution of the Stefan problem, it is useful to invert the inclusion

$$\chi \in \operatorname{sign}(\theta) \tag{2.1}$$

and to consider instead

$$\operatorname{sign}^{-1}(\chi) \ni \theta.$$

At this point, we added to the left hand side of the previous relation the term $\varepsilon \partial\chi/\partial t$, and we obtained the wanted estimates exploiting the monotonicity of the operator sign^{-1}.

So one might dispute that the inversion of the Stefan condition is due only to mathematical reasons, in such a way that the relaxation dynamics (1.1) can be put in the framework of the evolution equations studied in Theorem I.3.1 of Section I.3.

Then we could add the time derivative of the phase function directly to left hand side of (2.1) and obtain

$$\varepsilon \frac{\partial \chi}{\partial t} + \chi \in \operatorname{sign}(\theta) \qquad \text{in } Q. \tag{2.2}$$

Therefore we are led to consider the following version of the Stefan problem with phase relaxation:

$$\frac{\partial(\theta + \chi)}{\partial t} - \Delta\theta = f \qquad \text{in } Q, \tag{2.3}$$

$$\varepsilon \frac{\partial \chi}{\partial t} + \chi \in \operatorname{sign}(\theta) \qquad \text{in } Q \tag{2.4}$$

Now we show that th relation (2.2) can also be justified by means of a probabilistic interpretation of the phase transition that was given by Visintin in [38].

We assume that our physical system is composed by several small subsystems which we call *particles*. Moreover we suppose that any of these particles can assume either the solid state or the liquid state. This is in agreement with the concept of mushy region introduced in Section I.1. Let us call π_+ (respectively π_-) the probability of *melting a solid* (respectively *crystallizing a liquid*) particle in the unit time. Therefore we get that the melting rate per unit volume is proportional to $\pi_+(1-\chi)/2$ and the crystallizing rate per unit volume is proportional to $\pi_-(1+\chi)/2$. Hence

$$\partial_t \chi \text{ is proportional to } \pi_+(1-\chi)/2 - \pi_-(1+\chi)/2. \tag{2.5}$$

The transition probabilities above defined depend on the temperature, i.e. there exists a function $p : \mathbb{R} \longrightarrow [-1,1]$ such that we have two relations such $\pi_+ = p(\overline{\theta}^+)$ and $\pi_- = -p(-\overline{\theta}^-)$, where $\overline{\theta}^+ = \max\{\overline{\theta}, 0\}$ and $\overline{\theta}^- = \max\{-\overline{\theta}, 0\}$. Hence, the relation (2.5) means that there exists some constant $\varepsilon > 0$ such that

$$\varepsilon \partial_t \chi = \Big(p(\overline{\theta}^+) + p(\overline{\theta}^-)\Big) - \Big(p(\overline{\theta}^+) - p(\overline{\theta}^-)\Big)\chi. \tag{2.6}$$

Observe that the bigger $\partial_t \chi$ is, the smaller ε turns out to be. Equation (2.6) suggests the analysis of a relaxation dynamics like $\varepsilon \partial_t \chi = \phi(\overline{\theta}, \chi)$ for a suitable class of regular functions $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$. This is in fact the subject of paper [38], where L^1-tecniques are used.

Let us come back to (2.2). We take p equal to the single-valued sign_0 function, and then we allow it to be multivalued. In this way we obtain exactly our relaxation (2.2) and the constraint $-1 \leq \chi \leq 1$ is actually preserved (this can be easily deduced directly by (2.2)). This choice yields that every particle has the same probability, equal to 1, to change phase, however this does not seem to be a bad inconvenient when the rate of transition is very fast. Moreover this choice allows us to exploit L^2-tecniques only, and we are also able to replace the sign with a more genaral maximal monotone graph that is sublinear at infinity.

We set the problem in the same functional framework of the previous section, but we can slightly relax the assumption on the intial datum χ, we need in fact only

$$\chi_0 \in H. \tag{2.7}$$

Then the variational formulation problem (2.3)–(2.4) reads as follows.

Problem $(\mathbf{S2}_\varepsilon)$**.** Let $\varepsilon > 0$ and assume that (1.4)–(1.16), (2.7) hold. Find a pair $(\theta_\varepsilon, \chi_\varepsilon)$ satisfying the following conditions.

$$\theta_\varepsilon \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{2.8}$$

$$\theta_\varepsilon' \in L^2(0,T;V') + L^1(0,T;H), \tag{2.9}$$

$$\chi_\varepsilon \in H^1(0,T;H), \tag{2.10}$$

$$\exists \xi_\varepsilon \in L^2(Q), \quad \xi_\varepsilon \in \gamma(\theta_\varepsilon + \theta_D) \quad \text{ a.e. in } Q, \tag{2.11}$$

$$(\theta_\varepsilon + \chi_\varepsilon)' + A\theta_\varepsilon = f \quad \text{ in } V', \quad \text{a.e. in }]0,T[, \tag{2.12}$$

$$\varepsilon \chi_\varepsilon' + \chi_\varepsilon = \xi_\varepsilon \quad \text{ a.e. in } Q, \tag{2.13}$$

$$\theta_\varepsilon(0) = \theta_0, \quad \chi_\varepsilon(0) = \chi_0 \quad \text{ a.e. in } \Omega. \tag{2.14}$$

♠

In the sequel of this section, for convenience, in writing a solution of the Problem $(\mathbf{S2}_\varepsilon)$, we will omit the subscript ε. We begin with a uniqueness property.

Lemma II.2.1. *Let $\varepsilon > 0$. Problem* $(\mathbf{S2}_\varepsilon)$ *has at most one solution.*

Proof. For simplicity we omit the subscript ε. Let (θ_i, χ_i), $i = 1, 2$, two solutions to Problem $(\mathbf{S2}_\varepsilon)$ and let $\widetilde{\theta} := \theta_1 - \theta_2$ and $\widetilde{\chi} := \chi_1 - \chi_2$.

1st estimate

At first let us subtract the respective equations (2.12) for (θ_i, χ_i), $i = 1, 2$, from each other and integrate the difference from 0 to $s \in]0, t[$, $t \in [0, T]$. Then we test the result by $\widetilde{\theta}(s)$ and finally integrate over $]0, t[$. We get

$$\|\widetilde{\theta}\|^2_{L^2(0,t;H)} + \int_0^t \int_\Omega \widetilde{\chi}(s)\widetilde{\theta}(s)ds + \frac{1}{2}\|\nabla(I_0\widetilde{\theta})(t)\|^2_{\mathbf{H}} = 0. \tag{2.15}$$

2nd estimate

Now let us test equation (2.12) by $\varepsilon\widetilde{\theta}$ and integrate over $\Omega\times]0, t[$. We find

$$\frac{\varepsilon}{2}\|\widetilde{\theta}(t)\|^2_H + \varepsilon \int_0^t \int_\Omega \widetilde{\chi}'(s)\widetilde{\theta}(s)ds + \frac{\varepsilon}{2}\|\nabla(I_0\widetilde{\theta})\|^2_{L^2(0,t;\mathbf{H})} = 0. \tag{2.16}$$

3rd estimate

Let us multiply equation (2.13) by $\theta_\varepsilon = (\theta_1 + \theta_D) - (\theta_2 + \theta_D)$ and integrate in time and space. Thanks to the monotonicity of γ we find

$$0 \leq \varepsilon \int_0^t \int_\Omega \chi'_\varepsilon(s)\theta_\varepsilon(s)ds + \int_0^t \int_\Omega \chi_\varepsilon(s)\theta_\varepsilon(s)ds. \tag{2.17}$$

Rest of the proof

Adding (2.15), (2.16), and (2.17) we find that $\|\theta\|_{L^2(Q)} \leq 0$, i.e. $\theta_1 = \theta_2$ a.e. in Q. Thus it follows from a comparison in equation (2.12) that $\chi_1 = \chi_2$ a.e. in Q. $\square$

Now we construct a multivalued map Σ whose fixed point will give the solution of Problem $(\mathbf{S2}_\varepsilon)$ (see Definition A.4.1 for the definition of fixed points of a multivalued map). We will assume that (1.18) holds.

Let $\Theta \in L^2(0, T; H)$ and let us set

$$\Xi_\Theta := \left\{\xi \in L^2(Q) \ : \ \xi \in \gamma(\Theta + \theta_D) \text{ a.e. in } Q\right\}. \tag{2.18}$$

The set Ξ_Θ is nonempty. Now, for any $\xi \in \Xi_\Theta$ there exists one and only one $\chi_\xi \in H^1(0,T;H)$ such that

$$\varepsilon\chi_\xi' + \chi_\xi = \xi \qquad \text{a.e. in } Q, \tag{2.19}$$

$$\chi_\xi(0) = \chi_0 \qquad \text{a.e. in } \Omega \tag{2.20}$$

(observe that (2.19) is simply a linear ordinary differential equation). Now let t be arbitrarily chosen in $[0,T]$ and observe that multiplying equation (2.19) by χ_ξ', integrating in time and space, and applying Young inequality, we get, thanks to (2.20),

$$\frac{\varepsilon}{2}\|\chi_\xi'\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\chi_\xi(t)\|^2_H \le \frac{1}{2}\|\chi_0\|^2_H + \frac{1}{2\varepsilon}\|\xi\|^2_{L^2(0,t;H)}. \tag{2.21}$$

Now we estimate the right hand side of (2.21) using the assumption (1.18). We have that

$$\begin{aligned}
\|\xi\|^2_{L^2(0,t;H)} &= \int_0^t\int_\Omega |\xi(s)|^2 ds \\
&\le C_\gamma^2 \int_0^t\int_\Omega (1 + |\Theta(s) + \theta_D(s)|)^2\, ds \\
&\le 2C_\gamma^2 \int_0^t\int_\Omega \left(1 + |\Theta(s)|^2 + |\theta_D(s)|^2\right) ds \\
&\le 2C_\gamma^2 \left(1 + \|\theta_D\|^2_{L^2(0,t;H)} + \|\Theta\|^2_{L^2(0,t;H)}\right),
\end{aligned} \tag{2.22}$$

and therefore from (2.21)

$$\|\chi_\xi'\|^2_{L^2(0,t;H)} \le \frac{1}{\varepsilon}\|\chi_0\|^2_H + \frac{2C_\gamma^2}{\varepsilon^2}\left(1 + \|\theta_D\|^2_{L^2(0,t;H)} + \|\Theta\|^2_{L^2(0,t;H)}\right). \tag{2.23}$$

Then, since $\chi_\xi' \in L^2(0,T;H)$, by Theorem I.2.1 there exists a unique function θ_ξ such that

$$\theta_\xi \in L^2(0,T;V) \cap L^\infty(0,T;H) \tag{2.24}$$

$$\theta_\xi' \in L^2(0,T;V') + L^1(0,T;H) \tag{2.25}$$

$$\theta_\xi' + A\theta_\xi = f - \chi_\xi' \qquad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{2.26}$$

$$\theta_\xi(0) = \theta_0 \qquad \text{a.e. in } \Omega. \tag{2.27}$$

Proposition I.2.3 yields the following estimate.

$$\begin{aligned}
&\|\theta_\xi\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \\
&\le C\left(\|\theta_0\|_H + \|f\|_{L^2(0,T;V')+L^1(0,T;H)} + \|\chi_\xi'\|_{L^2(0,T;H)}\right),
\end{aligned} \tag{2.28}$$

where C is a positive constant depending only on T. Now set

$$\begin{aligned}&C(f,\theta_D,\theta_0,\chi_0)\\ &:= C\left(\|\theta_0\|_H^2 + \|f\|_{L^2(0,T;V')+L^1(0,T;H)} + \|\chi_0\|_H^2 + \|\theta_D\|^2_{L^2(0,T;H)}\right).\end{aligned} \tag{2.29}$$

Hence from (2.24), (2.28), and (2.23) we deduce that

$$\begin{aligned}\|\theta_\xi\|^2_{L^2(0,t;H)} &= \int_0^t \|\theta_\xi(s)\|_H^2 ds \le \int_0^t \|\theta_\xi\|^2_{L^\infty(0,t;H)} ds \le t\|\theta_\xi\|^2_{L^\infty(0,t;H)}\\ &\le tC(f,\theta_D,\theta_0,\chi_0) + tC\|\Theta\|^2_{L^2(0,t;H)}\\ &\le TC(f,\theta_D,\theta_0,\chi_0) + tC\|\Theta\|^2_{L^2(0,t;H)}\end{aligned} \tag{2.30}$$

for all $t \in [0,T]$.

Let us denote by $\Sigma(\Theta)$ the nonempty set of all functions θ_ξ satisfying (2.24)–(2.27), when ξ varies in Ξ_Θ and χ_ξ satisfies (2.19)–(2.20). Thus we have defined a multivalued operator $\Sigma : L^2(0,T;H) \longrightarrow \mathscr{P}(L^2(0,T;H))$. It is clear that a fixed point of Σ, i.e. a function $\theta \in L^2(0,T;H)$ such that $\theta \in \Sigma(\theta)$, is also a solution of Problem $(\mathbf{S2}_\varepsilon)$.

Proposition II.2.1. *Let $\varepsilon > 0$ and assume that* (1.18) *holds. Then Problem* $(\mathbf{S2}_\varepsilon)$ *admits a unique local solution, i.e. there exists $T_R \in [0,T]$ and there exists a unique pair (θ,χ) such that* (2.8)–(2.14) *hold with T replaced by T_R.*

Proof. Let $C(f,\theta_D,\theta_0,\chi_0)$ be the constant defined in (2.29) and set

$$R := 2TC(f,\theta_D,\theta_0,\chi_0).$$

Then define

$$T_R := \min\left\{\frac{1}{2C}, T\right\}$$

and

$$K := \left\{v \in L^2(0,T_R;H) \ : \ \|v\|_{L^2(0,T_R;H)} \le R^{1/2}\right\}. \tag{2.31}$$

By virtue of estimate (2.30) we have that the multivalued application Σ that we have defined above, maps K in $\mathscr{P}(K)$. We endow $L^2(0,T_R;H)$ with the weak topology, which is a Hausdorff locally convex vector topology that makes K a weakly compact set. Note also that K is metrizable, due to the separability of H. Our aim is to apply Theorem A.4.2 about fixed point of multivalued maps. First of all let us show that the reduced graph $G_{\mathcal{R}}(\Sigma) = \{(\theta,\Theta) \in K\times K \ : \ \theta \in \Sigma(\Theta)\}$ is weakly closed in $K \times K$. The fact that K is weakly compact and metrizable allows us to argue by sequences. Let $((\theta_m,\Theta_m))_{m\in\mathbb{N}_*}$ be a sequence in $G_{\mathcal{R}}(\Sigma)$

that weakly converges to a pair (θ, Θ). Then there exist sequences (χ_m) in $H^1(0,T_R;H)$ and (ξ_m) in $L^2(]0,T_R[\times\Omega)$ such that

$$\theta_m' + A\theta_m = f - \chi_m' \quad \text{in } V', \text{ a.e. in }]0,T_R[, \tag{2.32}$$
$$\varepsilon\chi_m' + \chi_m = \xi_m \quad \text{a.e. in }]0,T_R[\times\Omega, \tag{2.33}$$
$$\theta_m(0) = \theta_0, \quad \chi_m(0) = \chi_0 \quad \text{a.e. in } \Omega, \tag{2.34}$$
$$\xi_m \in \gamma(\Theta_m + \theta_D) \quad \text{a.e. in }]0,T_R[\times\Omega. \tag{2.35}$$

Since $\|\Theta_m\|_{L^2(0,T_R;H)}$ is bounded, estimate (2.23), which holds with χ and Θ replaced respectively by χ_m and Θ_m, $m \in \mathbb{N}_*$, implies that there exists $\chi \in H^1(0,T_R;H)$ such that, at least for a subsequence,

$$\chi_m \rightharpoonup \chi \quad \text{in } H^1(0,T_R;H). \tag{2.36}$$

Now observe that (2.28) holds with θ_ξ and χ_ξ replaced respectively by θ_μ and χ_μ, therefore from (2.36) and (2.32) and we deduce that $\theta_\mu' \rightharpoonup \theta'$ in $L^2(0,T_R;V')$, thus by the Aubin compactness lemma (cf. Theorem A.3.2) we infer the convergence

$$\theta_m \to \theta \quad \text{in } L^2(0,T_R;H). \tag{2.37}$$

Taking the limit in (2.32)–(2.35), then we find a function $\xi \in L^2(]0,T_R[\times\Omega)$ such that

$$\theta' + A\theta = f - \chi' \quad \text{in } V', \quad \text{a.e. in }]0,T_R[, \tag{2.38}$$
$$\varepsilon\chi' + \chi = \xi \quad \text{a.e. in }]0,T_R[\times\Omega, \tag{2.39}$$
$$\theta(0) = \theta_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.40}$$

Now, (2.37) yields that

$$(\xi_m, \Theta_m + \theta_D)_{L^2(]0,T_R[\times\Omega)} \to (\xi, \Theta + \theta_D)_{L^2(]0,T_R[\times\Omega)}$$

as $m \to \infty$. Therefore, since (2.35) holds, by virtue of Lemma I.3.1 we deduce that $\xi \in \gamma(\theta + \theta_D)$ a.e. in $]0,T_R[\times\Omega$, which together with (2.38)–(2.40), let us infer that $(\theta,\Theta) \in G_{\mathcal{R}}(\Sigma)$. It remains to prove that if $\Theta \in K$, then the set $\Sigma(\Theta)$ is closed and convex in K. To this aim let $\xi \in \Xi(\Theta)$. Concerning convexity, if $\theta_1, \theta_2 \in \Sigma(\Theta)$, then there exist $\xi_1, \xi_2 \in L^2(]0,T_R[\times\Omega)$ and $\chi_1, \chi_2 \in H^1(0,T_R;H)$ such that (2.24)–(2.27) hold with θ_ξ and χ_ξ replaced respectively by θ_i and χ_i, $i = 1,2$, $\xi_i \in \gamma(\Theta + \theta_D)$ a.e. in $]0,T_R[\times\Omega$, $i = 1,2$, and

$$\varepsilon\big((1-\mu)\chi_1 + \mu\chi_2\big)' + (1-\mu)\chi_1 + \mu\chi_2 = (1-\mu)\xi_1 + \mu\xi_2 \qquad \text{a.e. in }]0,T_R[\times\Omega, \tag{2.41}$$
$$(1-\lambda)\theta_1(0) + \lambda\theta_2(0) = \theta_0, \quad (1-\lambda)\chi_1(0) + \lambda\chi_2(0) = \chi_0 \quad \text{a.e. in } \Omega \tag{2.42}$$

for all $\mu \in]0,1[$. The maximal monotonicity of γ implies that $\gamma(r)$ is convex for all $r \in \mathbb{R}$, then $(1-\mu)\xi_1+\mu\xi_2 \in \gamma(\theta_X+u)$ a.e. in $]0,T_R[\times\Omega$. Then the linearity of equation (2.26) together with (2.42)–(2.42), implies that $(1-\mu)\theta_1+\mu\theta_2 \in \Sigma(\Theta)$ and we have shown that $\Sigma(\Theta)$ is convex. The proof of the closure of $\Sigma(\Theta)$ uses arguments that are similar to those employed to show the closure of $G_{\mathcal{R}}(\Sigma)$ and is actually much simpler. Now we can apply Theorem A.4.2 and deduce, also thanks to Lemma II.2.1, that there exists a unique pair of functions (θ,χ) that is a solution to Problem $(\mathbf{S2}_\varepsilon)$ on the interval $]0,T_R[$. □

Now we can present the main result of this section.

Theorem II.2.1. *Let $\varepsilon > 0$ and assume that* (1.18) *holds. Then Problem* $(\mathbf{S2}_\varepsilon)$ *admits a unique solution.*

Proof. Now we show that the local solution found in the previous Proposition can be extended to the whole interval $]0,T[$. To this aim we have to prove that the norm of any global solution can be controlled by a constant which is independent of $t \in [0,T[$. If (θ,χ) is such a solution, and if $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^2(0,T;H)$ are such that $f = f_{V'} + f_H$, then test (2.12) by θ. Then using Lemma I.4.5 we get that there exists a positive constant C, depending only on T, and $\mathscr{L}^d(\Omega)$ such that

$$\frac{1}{2}\|\theta(t)\|_H^2 + \int_0^t\int_\Omega \chi'(s)\theta(s)ds + \frac{1}{2}\|\nabla\theta\|^2_{L^2(0,t;\mathbf{H})}$$
$$\leq C + C\int_0^t \|\theta(s)\|_H^2 ds + \int_0^t \|f_H(s)\|_H^2\|\theta(s)\|_H^2 ds. \tag{2.43}$$

Now let ξ satisfy (2.11) and (2.13), and multiply equation (2.13) by χ' and integrate over $\Omega\times]0,t[$. Using Hölder and Young inequalities we infer that

$$\frac{\varepsilon}{2}\|\chi'\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\chi(t)\|_H^2 \leq \frac{1}{2}\|\chi_0\|_H^2 + \frac{1}{2\varepsilon}\|\xi\|^2_{L^2(0,t;H)}, \tag{2.44}$$

and arguing as in (2.22) in order to control the last term of (2.44) we get

$$\frac{\varepsilon}{2}\|\chi'\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\chi(t)\|_H^2 \leq C_\varepsilon + C_\varepsilon\int_0^t \|\theta(s)\|_H^2 ds, \tag{2.45}$$

where C_ε is a positive constant independent of θ, χ, ξ, and $t \in [0,T[$, but depending on ε. Now we add (2.43) and (2.45) and use Young inequality to treat the integral $\int_0^t\int_\Omega \chi'\theta$. We get a constant $C_\varepsilon > 0$, with the same dependencies as above, such that

$$\frac{1}{2}\|\theta(t)\|_H^2 + \frac{1}{2}\|\nabla\theta\|^2_{L^2(0,t;\mathbf{H})} + \frac{\varepsilon}{4}\|\chi'\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\chi(t)\|_H^2$$
$$\leq C_\varepsilon + C_\varepsilon \int_0^t \|\theta(s)\|_H^2 ds + \int_0^t \|f_H(s)\|_H^2\|\theta(s)\|_H^2 ds.$$

Now an application of the extended version of Gronwall Lemma given in Proposition A.5.2 yields the desired estimate. □

II.3 Convergence to the Stefan problem

This section is devoted to the analysis of the behaviour of the solution of $(\mathbf{S2}_\varepsilon)$ as the relaxation parameter goes to zero. The motivation of such asymptotic analysis is clear and it has been already illustrated in Section II.1 for the classical Stefan problem with phase relaxation.

We are going to recover some estimates, which are uniform with respect to ε and that will allow us to prove the desired convergence. We will make the non restrictive assumption that $\varepsilon < 1$. In order to perform the asymptotic limit we need to assume that

$$I_0 f \in L^2(0,T;H). \tag{3.1}$$

We recall that the notation $I_0 f$ is defined in Chapter I, formula (4.14).

Proposition II.3.1. *Let $\varepsilon > 0$. Assume* (1.18) *and* (3.1)*. Let $(\chi_\varepsilon, \theta_\varepsilon)$ be the solution to Problem $(\mathbf{S2}_\varepsilon)$ and let ξ_ε satisfying* (2.11) *and* (2.13)*. Then there exists a constant $C > 0$, independent of ε, such that*

$$\|\theta_\varepsilon\|_{L^2(0,T;H)} + \|I_0\theta_\varepsilon\|_{L^\infty(0,T;V)} + \varepsilon^{1/2}\|\theta\|_{L^\infty(0,T;H)} + \|\xi_\varepsilon\|_{L^2(0,T;H)}$$
$$+ \varepsilon^{1/2}\|\theta_\varepsilon\|_{L^2(0,T;V)} + \|\chi_\varepsilon\|_{L^2(0,T;H)} + \varepsilon^{1/2}\|\chi_\varepsilon\|_{L^\infty(0,T;H)} \leq C. \tag{3.2}$$

Proof. Let $t \in [0,T]$.

1st estimate

Let us integrate equation (2.12) over $]0,s[$, where $s \in [0,t]$, and let us test the result with the function $\theta_\varepsilon(s)$. Then, integrating over $]0,t[$ and using Lemma

I.4.5, we find a constant $C > 0$, independent of ε such that

$$\frac{1}{2}\|\theta_\varepsilon\|^2_{L^2(0,t;H)} + \int_0^t\int_\Omega \chi_\varepsilon(s)\theta_\varepsilon(s)ds + \frac{1}{4}\|\nabla(I_0\theta_\varepsilon)(t)\|^2_{\mathbf{H}}$$
$$\leq C + C\int_0^t \|\nabla(I_0\theta_\varepsilon)(s)\|^2_{\mathbf{H}}ds. \tag{3.3}$$

2^nd estimate

Now let us choose $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^1(0,T;H)$ such that $f = f_{V'} + f_H$. Let us test equation (2.12) by $\varepsilon\theta_\varepsilon$ and integrate over $\Omega\times]0,t[$. We find

$$\frac{\varepsilon}{2}\|\theta_\varepsilon(t)\|^2_H + \varepsilon\int_0^t\int_\Omega \chi'_\varepsilon(s)\theta_\varepsilon(s)ds + \frac{\varepsilon}{2}\|\nabla\theta_\varepsilon\|^2_{L^2(0,t;\mathbf{H})}$$
$$\leq C + C\int_0^t \varepsilon\|\theta_\varepsilon(s)\|^2_H ds + \int_0^t \|f_H(s)\|_H\varepsilon\|\theta_\varepsilon(s)\|_H ds, \tag{3.4}$$

for some positive constant C independent of ε.

3^rd estimate

Let us multiply equation (2.13) by θ_ε and integrate in time and space. We find

$$\varepsilon\int_0^t\int_\Omega \chi'_\varepsilon(s)\theta_\varepsilon(s)ds + \int_0^t\int_\Omega \chi_\varepsilon(s)\theta_\varepsilon(s)ds = \int_0^t\int_\Omega \xi_\varepsilon(s)\theta_\varepsilon(s)ds.$$

Now let $s_0 := \max\{|s| \ : \ s \in \gamma(0)\}$. Thanks to the monotonicity and to sublinearity of γ, we can write

$$\begin{aligned}
\int_0^t\int_\Omega \xi_\varepsilon(s)\theta_\varepsilon(s)ds &= \int_0^t\int_\Omega (\xi_\varepsilon(s) - s_0)\theta_\varepsilon(s)ds + \int_0^t\int_\Omega s_0\theta_\varepsilon(s)ds \\
&= \int_0^t\int_\Omega (\xi_\varepsilon(s) - s_0)(\theta_\varepsilon(s) + \theta_D(s))ds \\
&\quad - \int_0^t\int_\Omega (\xi_\varepsilon(s) - s_0)\theta_D(s)ds + \int_0^t\int_\Omega s_0\theta_\varepsilon(s)ds \\
&\geq - \int_0^t\int_\Omega (\xi_\varepsilon(s) - s_0)\theta_D(s)ds + \int_0^t\int_\Omega s_0\theta_\varepsilon(s)ds \\
&\geq -C_\gamma\int_0^t\int_\Omega (1 + |\theta_\varepsilon(s)| + |s_0|)|\theta_D(s)|ds \\
&\quad + \int_0^t\int_\Omega s_0\theta_\varepsilon(s)ds.
\end{aligned}$$

Hence we find a constant $C > 0$ depending only on the data of the problem, but independent of ε, such that

$$\begin{aligned} 0 \leq \varepsilon \int_0^t \int_\Omega \chi_\varepsilon'(s)\theta_\varepsilon(s)ds + \int_0^t \int_\Omega \chi_\varepsilon(s)\theta_\varepsilon(s)ds \\ + C + C \int_0^t \int_\Omega |\theta_\varepsilon(s)|(|s_0| + |\theta_D(s)|)ds. \end{aligned} \tag{3.5}$$

Therefore adding this inequality to (3.3) and (3.4) there are two cancellations, and applying the Hölder and Young inequalities to treat the last integral in (3.5), we find

$$\begin{aligned} &\frac{1}{4}\|\theta_\varepsilon\|^2_{L^2(0,t;H)} + \frac{1}{4}\|\nabla(I_0\theta_\varepsilon)(t)\|^2_{\mathbf{H}} + \frac{\varepsilon}{2}\|\theta_\varepsilon(t)\|^2_H + \frac{\varepsilon}{2}\|\nabla\theta_\varepsilon\|^2_{L^2(0,t;\mathbf{H})} \\ &\leq C + C\int_0^t \left(\|\nabla(I_0\theta_\varepsilon)(s)\|^2_{\mathbf{H}} + \varepsilon\|\theta_\varepsilon(s)\|^2_H\right) ds + \int_0^t \|f_H(s)\|_H \varepsilon \|\theta_\varepsilon(s)\|_H ds, \end{aligned}$$

therefore an application of the generalized Gronwall Lemma yields

$$\|\theta_\varepsilon\|^2_{L^2(0,t;H)} + \|\nabla(I_0\theta_\varepsilon)\|^2_{L^\infty(0,t;\mathbf{H})} + \varepsilon\|\theta_\varepsilon\|^2_{L^\infty(0,t;H)} + \varepsilon\|\nabla\theta_\varepsilon\|^2_{L^2(0,t;\mathbf{H})} \leq C. \tag{3.6}$$

4^{th} estimate

Finally we multiply the equation (2.13) by χ and integrate over $\Omega\times]0,t[$. Thanks to Hölder and Young inequalities we infer that

$$\frac{\varepsilon}{2}\|\chi_\varepsilon(t)\|^2_H + \frac{1}{2}\|\chi_\varepsilon\|^2_{L^2(0,t;H)} = \frac{\varepsilon}{2}\|\chi_0\|^2_H + \frac{1}{2}\|\xi_\varepsilon\|^2_{L^2(0,t;H)}.$$

Last term of the previous inequality can be treated using again the sublinearity of γ. Hence we find

$$\frac{\varepsilon}{2}\|\chi_\varepsilon(t)\|^2_H + \frac{1}{2}\|\chi_\varepsilon\|^2_{L^2(0,t;H)} \leq C + C\|\theta_\varepsilon\|^2_{L^2(0,t;H)},$$

for some positive constant $C > 0$ independent of ε. But the quantity $\|\theta_\varepsilon\|^2_{L^2(0,t;H)}$ was already bounded in (3.6), therefore we have (3.2). □

Now we are ready to show that the solution of Problem $(\mathbf{S2}_\varepsilon)$ converges in a suitable topology to the solution of the Stefan problem. In fact we can prove the following theorem.

Theorem II.3.1. *Assume* (1.18) *and* (3.1)*. For any $\varepsilon > 0$ let $(\theta_\varepsilon, \chi_\varepsilon)$ be the solution of Problem* **(S2$_\varepsilon$)**. *Moreover let (θ, χ) be the solution to Problem* **(S)**. *Then we have*

$$\theta_\varepsilon \rightharpoonup \theta \quad \text{in } L^2(0,T;H), \tag{3.7}$$

$$I_0\theta_\varepsilon \overset{*}{\rightharpoonup} I_0\theta \quad \text{in } L^\infty(0,T;V), \tag{3.8}$$

$$\chi_\varepsilon \rightharpoonup \chi \quad \text{in } L^2(0,T;H) \tag{3.9}$$

as $\varepsilon \searrow 0$.

Proof. Let us start by observing that an integration in time of (2.12) yields

$$\theta_\varepsilon + \chi_\varepsilon + A(I_0\theta_\varepsilon) = \theta_0 + \chi_0 + I_0 f \quad \text{in } V', \text{ in } [0,T]. \tag{3.10}$$

By Proposition II.3.1 we have that there exist three functions θ, χ, and ξ such that, at least for subsequences,

$$\theta_\varepsilon \rightharpoonup \theta \quad \text{in } L^2(0,T;H), \tag{3.11}$$

$$\varepsilon\theta_\varepsilon \to 0 \quad \text{in } L^2(0,T;V) \cap L^\infty(0,T;H), \tag{3.12}$$

$$I_0\theta_\varepsilon \overset{*}{\rightharpoonup} I_0\theta \quad \text{in } L^\infty(0,T;V) \cap H^1(0,T;H), \tag{3.13}$$

$$\chi_\varepsilon \rightharpoonup \chi \quad \text{in } L^2(Q), \tag{3.14}$$

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{in } L^2(Q). \tag{3.15}$$

Let us note that thanks to (3.2), a comparison in (2.13) yields that the sequence $\varepsilon\chi_\varepsilon'$ is bounded in $L^2(Q)$. Therefore there is a function $\zeta \in L^2(Q)$ such that $\varepsilon\chi_\varepsilon' \rightharpoonup \zeta$ in $L^2(Q)$. On the other hand from (3.14) we infer that $\chi_\varepsilon \to \chi$ and $\chi_\varepsilon' \to \chi'$ in $\mathscr{D}'(Q)$. Thus $\varepsilon\chi_\varepsilon' \to 0$ in $\mathscr{D}'(Q)$. Hence by the uniqueness of the limit we deduce that $\zeta = 0$ and then

$$\varepsilon\chi_\varepsilon' \rightharpoonup 0 \quad \text{in } L^2(Q). \tag{3.16}$$

Therefore, taking the limit in (3.10) and in (2.12) as $\varepsilon \searrow 0$, we find that $\chi = \xi$, that (1.26)–(1.28) and (1.31) are satisfied and that

$$\theta + \chi + A(I_0\theta) = \theta_0 + \chi_0 + I_0 f \quad \text{in } V', \text{ in } [0,T]. \tag{3.17}$$

$$\chi = \xi \quad \text{a.e. in } Q.$$

From convergence (3.13) it is easily seen that

$$I_0\theta_\varepsilon(t) \rightharpoonup I_0\theta(t) \quad \text{in } H, \ \forall t \in [0,T]. \tag{3.18}$$

Moreover applying Proposition A.2.5 we get that $I_0\theta_\varepsilon$ and $I_0\theta$ are weakly continuous functions from $[0,T]$ in V, therefore it makes sense to consider $(I_0\theta_\varepsilon)(t) \in V$

and $(I_0\theta)(t) \in V$ for any $t \in [0,T]$. Now, since the sequance $I_0\theta_\varepsilon$ is bounded in $L^\infty(0,T;V)$, we have that for all t there is a function $\eta_t \in V$ such that $(I_0\theta_\varepsilon)(t) \rightharpoonup \eta_t$ in $L^\infty(0,T;V)$. Hence from (3.18) and from the uniqueness of the limit we get that $\eta_t = (I_0\theta)(t)$ for every $t \in [0,T]$ and the following convergence holds true

$$I_0\theta_\varepsilon(t) \rightharpoonup I_0\theta(t) \qquad \text{in } V,\ \forall t \in [0,T]. \tag{3.19}$$

Now we prove the nonlinear relation (1.30). Let us notice that we have, thanks to (2.13),

$$\int_Q \xi_\varepsilon\theta_\varepsilon = \int_Q \varepsilon\chi_\varepsilon'\theta_\varepsilon + \int_Q \chi_\varepsilon\theta_\varepsilon. \tag{3.20}$$

Moreover, since $I_0 f \in L^2(0,T;H)$, by a comparison in (3.10) and in (3.17), we get that $A(I_0\theta_\varepsilon)$ and $A(I_0\theta)$ belong to $L^2(0,T;H)$, thus (3.10) and (3.17) can be read as equations in H, for any $t \in [0,T]$ fixed, i.e.

$$\begin{aligned} &\theta_\varepsilon + \chi_\varepsilon + A(I_0\theta_\varepsilon) = \theta_0 + \chi_0 + I_0 f \qquad \text{in } H,\ \text{in } [0,T],\\ &\theta + \chi + A(I_0\theta) = \theta_0 + \chi_0 + I_0 f \qquad \text{in } H,\ \text{in } [0,T]. \end{aligned} \tag{3.21}$$

Therefore we can write

$$\begin{aligned} \int_Q \chi_\varepsilon\theta_\varepsilon &= \int_0^T \Big(\theta_0 + \chi_0 + (I_0 f)(t) - \theta_\varepsilon(t) - A(I_0\theta_\varepsilon)(t), \theta_\varepsilon(t)\Big)_H dt\\ &= \int_0^T \Big(\theta_0 + \chi_0 + (I_0 f)(t), \theta_\varepsilon(t)\Big)_H dt - \|\theta_\varepsilon\|^2_{L^2(Q)} - \frac{1}{2}\|\nabla(I_0\theta_\varepsilon)(T)\|^2_{\mathbf{H}}. \end{aligned}$$

Therefore the convergences (3.11) and (3.19) and the lower semicontinuity of the norms imply that

$$\begin{aligned} &\limsup_{\varepsilon\searrow 0} \int_Q \chi_\varepsilon\theta_\varepsilon\\ &\le \int_0^T \Big((I_0 f)(t) + \theta_0 + \chi_0, \theta(t)\Big)_H dt - \|\theta\|^2_{L^2(Q)} - \frac{1}{2}\|\nabla(I_0\theta)(T)\|^2_{\mathbf{H}}\\ &= \int_0^T \Big(\theta_0 + \chi_0 + (I_0 f)(t) - \theta_\varepsilon(t) - A(I_0\theta_\varepsilon)(t), \theta_\varepsilon(t)\Big)_H dt = \int_Q \chi\theta \end{aligned} \tag{3.22}$$

(see Proposition A.2.6). Moreover we have, using (2.12), that

$$
\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\int_Q \varepsilon\chi_\varepsilon'\theta_\varepsilon \\
&= \limsup_{\varepsilon\searrow 0}\left(\int_0^T \langle\varepsilon f(t),\theta_\varepsilon(t)\rangle dt - \int_0^T \langle\varepsilon\theta_\varepsilon'(t),\theta_\varepsilon(t)\rangle dt - \int_0^T \langle\varepsilon A\theta_\varepsilon(t),\theta_\varepsilon(t)\rangle dt\right) \\
&= \limsup_{\varepsilon\searrow 0}\left(\int_0^T \langle f(t),\varepsilon\theta_\varepsilon(t)\rangle dt - \frac{\varepsilon}{2}\|\theta_\varepsilon(T)\|_H^2 + \frac{\varepsilon}{2}\|\theta_0\|_H^2 - \varepsilon\|\nabla\theta_\varepsilon\|^2_{L^2(0,T;\mathbf{H})}\right) \\
&\le \limsup_{\varepsilon\searrow 0}\left(\int_0^T \langle f(t),\varepsilon\theta_\varepsilon(t)\rangle dt + \frac{\varepsilon}{2}\|\theta_0\|_H^2\right) = 0. \qquad (3.23)
\end{aligned}
$$

Hence, collecting (3.20), (3.22), and (3.23), we deduce that

$$
\limsup_{\varepsilon\searrow 0}\int_Q \xi_\varepsilon(\theta_\varepsilon+\theta_D) \le \int_Q \chi(\theta+\theta_D). \qquad (3.24)
$$

Thanks to (2.11) and to Lemma I.3.1 this inequality yields (1.30). Thus (θ,χ) satisfies (1.26)–(1.28), (1.31) and

$$
\theta+\chi+A(I_0\theta) = \theta_0+\chi_0+I_0 f \quad \text{ in } V', \text{ in } [0,T]. \qquad (3.25)
$$

$$
\chi\in\gamma(\theta+\theta_D) \quad \text{ a.e. in } Q. \qquad (3.26)
$$

Now we prove that (θ,χ) is the unique pair satisfying (1.26)–(1.28), (1.31), (3.25), and (3.26). Then it follows that (θ,χ) is also the unique solution to Problem **(S)**, because any solution to **(S)** is also a solution to the "integrated" problem (5.1)–(5.3), (5.6), (3.25), (3.26). This also allows us to deduce that the whole sequences (θ_ε) and (χ_ε) converge. Let (θ_i,χ_i), $i=1,2$, be two pairs satisfying the above cited conditions. Set $\widetilde{\theta} := \theta_1-\theta_2$ and $\widetilde{\chi} := \chi_1-\chi_2$. Then taking the difference of the equations (3.25) written for (θ_1,χ_1) and (θ_2,χ_2), we find

$$
I_0\widetilde{\theta} \in L^2(0,T;V)\cap H^1(0,T;H), \qquad (3.27)
$$

$$
\widetilde{\chi} \in L^2(0,T;H), \qquad (3.28)
$$

$$
\widetilde{\theta}+\widetilde{\chi}+A(I_0\widetilde{\theta}) = 0 \quad \text{ in } V', \text{ in }]0,T[, \qquad (3.29)
$$

$$
\widetilde{\chi} \in \gamma(\theta_1+u)-\gamma(\theta_2+u) \quad \text{ a.e. in } Q. \qquad (3.30)
$$

Since $A(I_0\widetilde{\theta}) \in L^2(0,T;H)$, we have that

$$
\widetilde{\theta}+\widetilde{\chi}+A(I_0\widetilde{\theta}) = 0 \quad \text{ in } H, \text{ in }]0,T[. \qquad (3.31)
$$

Multiplying (3.31) by $\widetilde{\theta}$ and integrating over $\Omega\times]0,t[$, we get

$$\|\widetilde{\theta}\|^2_{L^2(0,t;H)} + \int_0^t \int_\Omega \widetilde{\chi}(s)\widetilde{\theta}(s)ds + \frac{1}{2}\|\nabla(I_0\widetilde{\theta})(t)\|^2_{H^n} = 0. \tag{3.32}$$

Therefore, since $\widetilde{\chi}\widetilde{\theta} \geq 0$ a.e. in Q, we infer that $\widetilde{\theta} = 0$ a.e. in Q and, by a comparison in (3.31), $\widetilde{\chi} = 0$ a.e. in Q. □

Remark II.3.1. System (2.11)–(2.13) may also be regarded as the variational inequality

$$\langle\theta'_\varepsilon - \frac{1}{\varepsilon}\chi_\varepsilon + A\theta_\varepsilon - f, v - \theta_\varepsilon - u\rangle + \frac{1}{\varepsilon}\int_\Omega (\psi(v) - \psi(\theta_\varepsilon + u)) \geq 0$$
$$\forall v \in V, \text{ a.e. in }]0,T[, \tag{3.33}$$

where $\psi : \mathbb{R} \to [0,\infty]$ is a proper, convex, lower semicontinuos function such that $\gamma = \partial\psi$.

Remark II.3.2. If we solve the ordinary differential equation (2.13) with the second initial condition of (2.14), we obtain for χ_ε the following expression

$$\chi_\varepsilon(t) = \varepsilon\psi_\varepsilon(t)\chi_0 + (\psi_\varepsilon * \xi_\varepsilon)(t), \tag{3.34}$$

where we set $\psi_\varepsilon(t) := \varepsilon^{-1}e^{-t/\varepsilon}$, and where the symbol " $*$ " denotes the usual convolution product with respect to time, that is $(a*b)(t) := \int_0^t a(t-s)b(s)ds,\ t \in [0,T]$, a and b being functions that may also depend on the space variables. Hence, using (2.12)–(2.13), we are led to the system

$$(\theta_\varepsilon + \psi_\varepsilon * \xi_\varepsilon)' + A\theta_\varepsilon = f_\varepsilon \ \text{ in } V', \ \text{ a.e. in }]0,T[, \tag{3.35}$$
$$\xi_\varepsilon \in \gamma(\theta_\varepsilon + u) \ \text{ a.e. in } Q, \tag{3.36}$$

with $f_\varepsilon(t) := f(t) + \psi_\varepsilon(t)\chi_0$. Similar problems are dealt with in [12] and this suggests that the tecniques used in that paper might be exploited to solve our Problem $(\mathbf{S2}_\varepsilon)$.

Chapter III

Cattaneo-Maxwell heat flux law

This chapter is devoted to the analysis of the relaxed Stefan problem with Cattaneo-Maxwell heat flux law. In the first section we introduce the heat conduction law by Cattaneo and in the second section we prove an existence result for the relaxed Stefan problem where the Cattaneo law has been assumed. We point out that uniqueness of solutions for such model is still an open problem. Finally in the last section we prove that the solutions of the relaxed system converge in a suitable topology to the solution of the Stefan problem, when the relaxation parameters tend independently to zero.

III.1 The relaxed hyperbolic Stefan problem

The Stefan model studied so far is essentially based on the heat equation coupled with the Fourier law, i.e.

$$\frac{\partial \theta}{\partial t} - \Delta\theta = 0 \qquad \text{in } Q = \Omega\times]0,T[. \tag{1.1}$$

The heat equation has a particular feature: if at the time $t = 0$ the absolute value of the temperature θ of some point x_0 is strictly positive and if at every other point is zero, then, at least for small times, $|\theta|$ is strictly positive at every point of Ω. This property can be derived from the so called *strong maximum principle* which essentially states that if the maximum of θ is attained at some point $(x_M, t_M) \in Q$, then θ is constant on $\Omega\times]0, t_M[$. We refer to [18, Theorem 11, Section 7.1] for the proof of this principle. This feature is usually rephrased by saying that the heat equation supports *infinite propagation speed of disturbances*.

Now strong maximum principle is false for another type of partial differential equation, namely the well known *wave equation*

$$\frac{\partial^2 \theta}{\partial t^2} - \Delta\theta = 0 \qquad \text{in } Q = \Omega\times]0,T[$$

which models, for instance, the behaviour of a vibrating string or more generally an elastic body. In fact it can be proved that if θ and $\partial\theta/\partial t$ are identically zero on a ball B of radius t_0 centered in x_0 ($(x_0, t_0) \in Q$), then $\theta \equiv 0$ within the cone $C := \{0 \leq t \leq t_0,\ |x - x_0| \leq t_0 - t\}$. Therefore if any disturbance originates outside B, then it does not affect the solution θ within C. This phenomenon is at variance with the case of heat equation, and indeed it usually said that *initial disturbances propagate at finite speed* (see again [18, Theorem 8, Section 7.2] for these arguments).

Now one can argue that heat is instead expected to propagate with a finite speed, so that a change in the model given by (1.1) seems mandatory. On the other hand, from a macroscopic point of view it is also reasonable to assume that the infinite propagation of heat is a good approximation of real phenomena, and the heat equation is sufficient to describe the thermal evolution of a physical system.

Moving from these considerations a former approach to modify the classical model is due to Carlo Cattaneo, who in 1948 (cf. [7]) proposed to replace the Fourier law

$$\mathbf{q} = -\nabla\theta \qquad \text{in } Q,$$

by an alternative model of heat conduction based on the so called *Cattaneo-Maxwell heat flux law*:

$$\alpha\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\nabla\theta \qquad \text{in } Q, \tag{1.2}$$

where $\mathbf{q}$ is the heat flux. In fact, the constituive assumption (1.2) appeared for the first time in 1867 in a paper by Maxwell (see [25]), but he neglected the term $\alpha\partial\mathbf{q}/\partial t$ because "the rate of conduction will rapidly establish itself".

Now observe that a formal integration of (1.2) gives

$$\mathbf{q}(t) = -\frac{1}{\alpha}\int_0^t \exp\left(\frac{s-t}{\alpha}\right)\nabla\theta(s)ds \qquad \text{in } Q, \tag{1.3}$$

therefore coupling (1.3) and the relation

$$\frac{\partial\theta}{\partial t} + \operatorname{div}\mathbf{q} = g \qquad \text{in } Q$$

governing the evolution of the temperature, we find

$$\frac{\partial\theta}{\partial t} - \frac{1}{\alpha}\int_0^t \exp\left(\frac{s-t}{\alpha}\right)\Delta\theta(s)ds = g \qquad \text{in } Q.$$

Hence differentiating in time the last equation we get

$$\frac{\partial^2\theta}{\partial t^2} - \Delta\theta + \frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{1}{\alpha}g + \frac{\partial g}{\partial t} \qquad \text{in } Q. \tag{1.4}$$

Now, (1.4) belongs to the class of *hyperbolic equations* (see [16] for classification of PDEs), and for this equations, like for the wave equation, it can be proved that initial disturbances have finite speed of propagation (see again [18]).

In this chapter we want to study the model proposed by Cattaneo in the framework of the phase transition models, therefore we consider a further generalization of the Stefan model which takes account of undercooling effects and of finite propagation speed of thermal disturbances. Then the model is given by

$$\frac{\partial(\theta+\chi)}{\partial t} + \operatorname{div}\mathbf{q} = g \quad \text{in } Q, \tag{1.5}$$

$$\alpha\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\nabla\theta \quad \text{in } Q, \tag{1.6}$$

$$\varepsilon\frac{\partial \chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta \quad \text{in } Q. \tag{1.7}$$

This model is also called *hyperbolic Stefan problem with phase relaxation* and we supply it with the following initial and boundary conditions:

$$\theta = \theta_D \quad \text{on } \Gamma_D \times]0,T[, \tag{1.8}$$

$$\mathbf{q}\cdot\mathbf{n} = \varphi_N \quad \text{on } \Gamma_N \times]0,T[, \tag{1.9}$$

$$\theta(\cdot,0) = \theta_0, \quad \chi(\cdot,0) = \chi_0, \quad \mathbf{q}(\cdot,0) = \mathbf{q}_0 \quad \text{on } \Omega. \tag{1.10}$$

here Γ_0 and Γ_1 denote two measurable subsets in which the boundary of Ω is partitioned, then $\theta_D, \varphi_N, \theta_0, \chi_0, \mathbf{q}_0$ are given functions. We assume that θ_D is a sufficiently smooth function defined on the whole Q and that there exists a vector function $\mathbf{q}_N : Q \to \mathbb{R}^n$ such that $\mathbf{q}_N \cdot \mathbf{n} = \varphi_N$ on $\Gamma_1 \times (0,T)$ in a suitable sense. Hence, setting $\overline{\theta}_0 := \theta_0 - \theta_D(0)$ and $\overline{\mathbf{q}}_0 := \mathbf{q}(0) - \mathbf{q}_N(0)$, we are led to consider the following system in the new unknown $\overline{\theta} := \theta - \theta_D$, χ, and $\overline{\mathbf{q}} := \mathbf{q} - \mathbf{q}_N$:

$$\frac{\partial(\overline{\theta}+\chi)}{\partial t} + \operatorname{div}\mathbf{q} = g - \frac{\partial\theta_D}{\partial t} - \operatorname{div}\mathbf{q}_N \quad \text{in } Q, \tag{1.11}$$

$$\alpha\frac{\partial \overline{\mathbf{q}}}{\partial t} + \overline{\mathbf{q}} = -\nabla\overline{\theta} - \nabla\theta_D - \alpha\frac{\partial \mathbf{q}_N}{\partial t} - \mathbf{q}_N \quad \text{in } Q, \tag{1.12}$$

$$\varepsilon\frac{\partial \chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \overline{\theta} + \theta_D \quad \text{in } Q, \tag{1.13}$$

$$\overline{\theta} = 0 \quad \text{on } \Gamma_D \times]0,T[, \quad \overline{\mathbf{q}}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N \times]0,T[, \tag{1.14}$$

$$\overline{\theta}(\cdot,0) = \theta_0, \quad \chi(\cdot,0) = \chi_0, \quad \overline{\mathbf{q}}(\cdot,0) = \mathbf{q}_0 \quad \text{in } \Omega. \tag{1.15}$$

This formulation is convenient because we have homogeneous boundary conditions for $\overline{\theta}$ and $\overline{\mathbf{q}}$. In the next section, performing a rigorous analysis of such problem, we will denote by f the right hand side of (1.11), and we will set

$$\mathbf{h}_\alpha := -\nabla\theta_D - \alpha\frac{\partial \mathbf{q}_N}{\partial t} - \mathbf{q}_N. \tag{1.16}$$

As we mentioned before, the relaxation parameters α and ε are usually very small with respect to the used length scale, and often the Stefan problem is therefore considered as an approximation of the relaxed system. Hence it is natural to wonder whether the solutions of the hyperbolic Stefan problem with phase relaxation converge to the solution of the Stefan problem

$$\frac{\partial(\overline{\theta}+\chi)}{\partial t} - \Delta\overline{\theta} = g - \frac{\partial\theta_D}{\partial t} - \operatorname{div}\mathbf{q}_N \quad \text{in } Q, \tag{1.17}$$

$$\chi \in \operatorname{sign}(\overline{\theta}+\theta_D) \quad \text{in } Q, \tag{1.18}$$

$$\overline{\theta} = 0 \quad \text{on } \Gamma_D\times]0,T[, \quad \partial_{\mathbf{n}}\overline{\theta} = 0 \quad \text{on } \Gamma_N\times]0,T[, \tag{1.19}$$

$$(\overline{\theta}+\chi)(\cdot,0) = \theta_0 + \chi_0, \quad \text{in } \Omega. \tag{1.20}$$

In the subsequent sections we prove that such convergence actually holds in a suitable topology. Let us remark also that anyway there could be some materials where the Fourier law does not seem to be very satisfactory, so that the Cattaneo law appears really necessary. We refer to the paper [10] for this kind of problems and to [8] and [28] for updated reviews of Cattaneo theory.

III.2 Existence

Within this section we show that the relaxed hyperbolic Stefan problem admits at least one weak solution in a variational setting. Let us now fix some notation. We set the problem in the same functional framework as the first chapter, however we recall all the data for convenience.

$$d \in \mathbb{N}_*, \quad \Omega \text{ is a bounded open and connected subset of } \mathbb{R}^d, \tag{2.1}$$

$$\Gamma := \partial\Omega \text{ is of Lipschitz class}, \tag{2.2}$$

$$\mathbf{n} \text{ is the outward normal unit vector to } \Omega, \tag{2.3}$$

$$\Gamma_D \text{ and } \Gamma_N \text{ are open subsets of } \Gamma \tag{2.4}$$

$$\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma, \quad \Gamma_D \cap \Gamma_N = \varnothing, \quad \overline{\Gamma}_D \cap \overline{\Gamma}_N \text{ is of Lipschitz class} \tag{2.5}$$

$$Q := \Omega\times]0,T[, \text{ where } T \in]0,\infty[. \tag{2.6}$$

Then we set

$$H := L^2(\Omega), \qquad V := H^1_{\Gamma_D}(\Omega), \tag{2.7}$$

endowed with their usual inner product. The inner product in H will be denoted by $(\cdot,\cdot)_H$, whereas we use the usual brackets ${}_{V'}\langle\cdot,\cdot\rangle_V$ for the duality pairing between V' and V. Moreover we set

$$\mathbf{H} := L^2(\Omega;\mathbb{R}^d), \tag{2.8}$$

whose usual inner product will be denoted by $(\cdot,\cdot)_{\mathbf{H}}$. We recall that (see Definition A.1.6):

$$L^2_{\mathrm{div}}(\Omega) := \{\mathbf{v} \in \mathbf{H} \ : \ \operatorname{div}\mathbf{v} \in H\}\,. \tag{2.9}$$

The space $L^2_{\mathrm{div}}(\Omega)$ is endowed with the inner product

$$(\mathbf{v}_1,\mathbf{v}_2)_{L^2_{\mathrm{div}}(\Omega)} := (\mathbf{v}_1,\mathbf{v}_2)_{\mathbf{H}} + (\operatorname{div}\mathbf{v}_1, \operatorname{div}\mathbf{v}_2)_H, \qquad \mathbf{v}_1,\mathbf{v}_2 \in L^2_{\mathrm{div}}(\Omega). \tag{2.10}$$

If $\mathbf{v} \in L^2_{\mathrm{div}}(\Omega)$, thanks to Theorem A.1.3 we know that $\mathbf{v}\cdot\mathbf{n} \in H^{-1/2}(\Gamma)$ and the restriction $\mathbf{v}\cdot\mathbf{n}|_{\Gamma_N}$ makes sense in $(H^{1/2}_{00}(\Gamma_N))'$ (cf. Remark A.1.1). Now we can introduce the closed subspace of $L^2_{\mathrm{div}}(\Omega)$

$$\mathbf{V} := \left\{\mathbf{v} \in L^2_{\mathrm{div}}(\Omega) : \ \mathbf{v}\cdot\mathbf{n}|_{\Gamma_N} = 0\right\}. \tag{2.11}$$

We use the notation ${}_{\mathbf{V}'}\langle\cdot,\cdot\rangle_{\mathbf{V}}$ for the duality pairing between $\mathbf{V}$ and $\mathbf{V}'$. By identifying $\mathbf{H}$ with its dual space, we get the Hilbert triplet

$$\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$$

with dense and continuous embeddings. Notice that these inclusions are not compact. We recall that the operator $A \in \mathscr{L}(V,V')$ is defined by

$${}_{V'}\langle Av_1, v_2\rangle_V := \int_\Omega \nabla v_1 \cdot \nabla v_2, \qquad v_1, v_2 \in V. \tag{2.12}$$

Moreover, we will consider the operators $B \in \mathscr{L}(\mathbf{H}, V')$ and $\mathbf{L} \in \mathscr{L}(H, \mathbf{V}')$ defined by

$${}_{V'}\langle B\mathbf{u}, v\rangle_V := -\int_\Omega \mathbf{u}\cdot\nabla v, \qquad \mathbf{u} \in \mathbf{H}, v \in V, \tag{2.13}$$

$${}_{\mathbf{V}'}\langle \mathbf{L}u, \mathbf{v}\rangle_{\mathbf{V}} := \int_\Omega u \operatorname{div}\mathbf{v}, \qquad u \in H, \mathbf{v} \in \mathbf{V}. \tag{2.14}$$

The following lemmas are easily proved by means of the Green formula.

Lemma III.2.1. *Let* $\mathbf{v}_0 \in \mathbf{H}$. *If there exists a function* $u_0 \in H$ *such that* $B\mathbf{v}_0 = u_0$, *i.e.,*

$${}_{V'}\langle B\mathbf{v}_0, v\rangle_V = -\int_\Omega \mathbf{v}_0\cdot\nabla v = \int_\Omega u_0 v \qquad \forall v \in V, \tag{2.15}$$

then $\mathbf{v}_0 \in \mathbf{V}$, $\operatorname{div}\mathbf{v}_0 = u_0$, *and* $\|\mathbf{v}_0\|_{\mathbf{V}} \le \|\mathbf{v}_0\|_{\mathbf{H}} + \|u_0\|_H$.

Lemma III.2.2. *Let $u_0 \in H$. If there is a function $\mathbf{v}_0 \in \mathbf{H}$ such that $\mathbf{L}u_0 = \mathbf{v}_0$, i.e.,*

$$
{}_{\mathbf{V}'}\langle \mathbf{L}u_0, \mathbf{v}\rangle_{\mathbf{V}} = \int_\Omega u_0 \operatorname{div} \mathbf{v} = \int_\Omega \mathbf{v}_0 \cdot \mathbf{v} \qquad \forall \mathbf{v} \in \mathbf{V}, \tag{2.16}
$$

then $u_0 \in V$, $\mathbf{v}_0 = -\nabla u_0$, and $\|u_0\|_V \le \|u_0\|_H + \|\mathbf{v}_0\|_{\mathbf{H}}$.

Now we list the assumptions on the data. We have

$$
\gamma : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R}) \quad \text{maximal monotone}, \qquad \beta := \alpha^{-1}, \tag{2.17}
$$

and we define $\varphi : \mathbb{R} \longrightarrow \widetilde{\mathbb{R}}$ by setting

$$
\varphi(r) := \begin{cases} \displaystyle\int_{r_0}^{r} \beta_0(s)ds & \text{if } r \in \overline{D(\beta)} \\ \infty & \text{if } r \notin \overline{D(\beta)} \end{cases}. \tag{2.18}
$$

Finally we have that

$$
f \in L^2(0,T;V') + L^1(0,T;H), \tag{2.19}
$$
$$
\mathbf{h} \in L^2(0,T;\mathbf{H}) \tag{2.20}
$$
$$
\mathbf{h}_\alpha \in L^2(0,T;\mathbf{H}), \qquad \mathbf{h}_\alpha \to \mathbf{h} \quad \text{in } L^2(0,T;\mathbf{H}) \text{ as } \alpha \searrow 0, \tag{2.21}
$$
$$
\theta_D \in H^1(0,T;H) \cap L^2(0,T;H^1(\Omega)), \tag{2.22}
$$
$$
\theta_0 \in H, \tag{2.23}
$$
$$
\mathbf{q}_0 \in \mathbf{H}, \tag{2.24}
$$
$$
\chi_0 \in H, \quad \varphi(\chi_0) \in L^1(\Omega). \tag{2.25}
$$

Remark III.2.1. Concernig the data presented in the previous section, we observe that the assumption on θ_D and the regularities $g \in L^1(0,T;L^2(\Omega)) \cap L^2(0,T;V')$ and $\mathbf{q}_N \in H^1(0,T;(L^2(\Omega))^n) \cap L^2(0,T;L^2_{\mathrm{div}}(\Omega))$ actually ensure that (2.19)–(2.22) hold.

Now we can present the weak formulation of problem (1.11)–(1.15)

Problem ($\mathbf{S1}_{\alpha\varepsilon}$). Let $\alpha > 0$ and $\varepsilon > 0$, and assume that (2.1)–(2.25) hold. Find a triplet $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ satisfying the following conditions.

$$\theta_{\alpha\varepsilon} \in L^\infty(0,T;H), \tag{2.26}$$
$$\theta'_{\alpha\varepsilon} \in L^2(0,T;V') + L^1(0,T;H), \tag{2.27}$$
$$\chi_{\alpha\varepsilon} \in H^1(0,T;H), \tag{2.28}$$
$$\exists \xi_\varepsilon \in L^2(Q), \quad \xi_{\alpha\varepsilon} \in \beta(\chi_{\alpha\varepsilon}) \quad \text{a.e. in } Q, \tag{2.29}$$
$$\mathbf{q}_{\alpha\varepsilon} \in H^1(0,T;\mathbf{V}') \cap L^\infty(0,T;\mathbf{H}), \tag{2.30}$$
$$(\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon})' + B\mathbf{q}_{\alpha\varepsilon} = f \quad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{2.31}$$
$$\alpha\mathbf{q}'_{\alpha\varepsilon} + \mathbf{q}_{\alpha\varepsilon} = \mathbf{L}\theta_{\alpha\varepsilon} + \mathbf{h}_\alpha \quad \text{in } \mathbf{V}', \quad \text{a.e. in }]0,T[, \tag{2.32}$$
$$\varepsilon\chi'_{\alpha\varepsilon} + \xi_{\alpha\varepsilon} \ni \theta_{\alpha\varepsilon} + \theta_D \quad \text{a.e. in } Q, \tag{2.33}$$
$$\theta_{\alpha\varepsilon}(0) = \theta_0 \quad \text{in } V', \qquad \chi_{\alpha\varepsilon}(0) = \chi_0 \quad \text{in } H, \qquad \mathbf{q}_{\alpha\varepsilon}(0) = \mathbf{q}_0 \quad \text{in } \mathbf{V}'. \tag{2.34}$$

♠

Within this section we show that Problem $(\mathbf{S1}_{\alpha\varepsilon})$ has at least one solution. To this aim we perform a regularization procedure by adding the term $\mu A\theta_{\alpha\varepsilon}$ to the left hand side of (2.31). Then, we solve the approximate problem by means of a fixed-point technique and finally take the limit as $\mu \searrow 0$. Of course it is not restrictive to suppose all the positive constants μ, α, and ε to be less than 1. For simplicity in dealing with the solutions of regularized problem, we will omit the subscript $\alpha\varepsilon$.

Lemma III.2.3. *Let $\mu > 0$ and assume that* (2.1)–(2.25) *hold. If a vector function $\mathbf{p}_\mu \in L^\infty(0,T;\mathbf{H})$ is given, then there exists a unique triplet $(\theta_\mu, \chi_\mu, \mathbf{q}_\mu)$ such that*

$$\theta_\mu \in L^2(0,T;V), \tag{2.35}$$
$$\theta'_\mu \in L^2(0,T;V') + L^1(0,T;H), \tag{2.36}$$
$$\chi_\mu \in H^1(0,T;H), \tag{2.37}$$
$$\mathbf{q}_\mu \in H^1(0,T;\mathbf{H}), \tag{2.38}$$
$$\exists \xi_\mu \in L^2(Q), \quad \xi_\mu \in \beta(\chi_\mu) \quad \textit{a.e. in } Q, \tag{2.39}$$
$$(\theta_\mu + \chi_\mu)' + \mu A\theta_\mu = f - B\mathbf{p}_\mu \quad \textit{in } V', \quad \textit{a.e. in }]0,T[, \tag{2.40}$$
$$\alpha\mathbf{q}'_\mu + \mathbf{q}_\mu = -\nabla\theta_\mu + \mathbf{h}_\alpha \quad \textit{a.e. in } Q, \tag{2.41}$$
$$\varepsilon\chi'_\mu + \xi_\mu \ni \theta_\mu + \theta_D \quad \textit{a.e. in } Q, \tag{2.42}$$
$$\theta_\mu(0) = \theta_0, \; \chi_\mu(0) = \chi_0, \; \mathbf{q}_\mu(0) = \mathbf{q}_0 \quad \textit{a.e. in } Q. \tag{2.43}$$

Proof. Since $\mathbf{p}_\mu \in L^\infty(0,T;\mathbf{H})$, it follows that there exists one and only one pair (θ_μ, χ_μ) satisfying (2.35)–(2.37), (2.40), (2.42), and the first two conditions of

(2.43). To see this, it is sufficient to apply Theorem I.4.1. Next, if we plug the found θ_μ into (2.41), it is an easy matter to prove that there exists a unique $\mathbf{q}_\mu$ satisfying (2.38), (2.41), and the initial condition in (2.43) (note that (2.41) is nothing but a linear ODE of first order). $\square$

Lemma III.2.3 defines a nonlinear operator $\Sigma_\mu : L^\infty(0,T;\mathbf{H}) \longrightarrow L^\infty(0,T;\mathbf{H})$ which maps $\mathbf{p}_\mu \in L^\infty(0,T;\mathbf{H})$ into the vector function $\mathbf{q}_\mu$, where $(\theta_\mu, \chi_\mu, \mathbf{q}_\mu)$ is the unique solution to (2.35)–(2.43). The next lemma will permit us to apply the Banach fixed point theorem.

Lemma III.2.4. *There exists a constant $C > 0$, depending only on T, ε, α, and μ, such that for any $\mathbf{p}_1, \mathbf{p}_2 \in L^\infty(0,T;\mathbf{H})$ there holds*

$$\|\Sigma_\mu(\mathbf{p}_1) - \Sigma_\mu(\mathbf{p}_2)\|^2_{L^\infty(0,t;\mathbf{H})} \le C \int_0^t \|\mathbf{p}_1 - \mathbf{p}_2\|^2_{L^\infty(0,s;\mathbf{H})} ds \qquad \forall t \in]0,T]. \tag{2.44}$$

Proof. For simplicity we omit the subscript μ. Let $(\theta_i, \chi_i, \mathbf{q}_i)$, $i = 1, 2$, be two triplets satisfying (2.35)–(2.43) with $\mathbf{p}$ replaced by $\mathbf{p}_i$, $i = 1, 2$. Set $\widetilde{\theta} := \theta_1 - \theta_2$, $\widetilde{\chi} := \chi_1 - \chi_2$, $\widetilde{\mathbf{p}} := \mathbf{p}_1 - \mathbf{p}_2$, $\widetilde{\mathbf{q}} := \mathbf{q}_1 - \mathbf{q}_2$ and let s, t satisfy $0 \le s \le t \le T$. First we integrate the difference of equations (2.40) from 0 to $\tau \in (0,s)$, test it by $\widetilde{\theta}(\tau)$, and integrate over $(0,s)$. Thanks to an integration by parts and using Hölder and Young inequalities, we have that

$$\begin{aligned}\int_0^s \int_\Omega (I_0\widetilde{\mathbf{p}})(\tau) \cdot \nabla\widetilde{\theta}(\tau) d\tau &= \int_\Omega (I_0\widetilde{\mathbf{p}})(s) \cdot \nabla(I_0\widetilde{\theta})(s) - \int_0^s \int_\Omega \widetilde{\mathbf{p}}(\tau) \cdot \nabla(I_0\widetilde{\theta})(\tau) d\tau \\ &\le \frac{\mu}{4} \|\nabla(I_0\widetilde{\theta})(s)\|^2_{\mathbf{H}} + \frac{1}{\mu} \|(I_0\widetilde{\mathbf{p}})(s)\|^2_{\mathbf{H}} + \frac{1}{2} \|\widetilde{\mathbf{p}}\|^2_{L^2(0,s;\mathbf{H})} \\ &\quad + \frac{\mu}{4} \int_0^s \|\nabla(I_0\widetilde{\theta})(\tau)\|^2_{\mathbf{H}} d\tau,\end{aligned}$$

with

$$\|(I_0\widetilde{\mathbf{p}})(s)\|^2_{\mathbf{H}} \le s \|\widetilde{\mathbf{p}}\|^2_{L^2(0,s;\mathbf{H})} \tag{2.45}$$

(recall Lemma I.4.4). Hence, we get

$$\begin{aligned}&\|\widetilde{\theta}\|^2_{L^2(0,s;H)} + \int_0^s \int_\Omega \widetilde{\chi}(\tau)\widetilde{\theta}(\tau) d\tau + \frac{\mu}{4} \|\nabla(I_0\widetilde{\theta})(s)\|^2_{\mathbf{H}} \\ &\le C \|\widetilde{\mathbf{p}}\|^2_{L^2(0,s;\mathbf{H})} + \frac{1}{2} \int_0^s \|\nabla(I_0\widetilde{\theta})(\tau)\|^2_{\mathbf{H}} d\tau.\end{aligned} \tag{2.46}$$

Multiplying the difference of the two equations (2.42) by $\widetilde{\chi}$, and integrating over $\Omega \times]0,s[$, the monotonicity of β implies that

$$\frac{\varepsilon}{2} \|\widetilde{\chi}(s)\|^2_H \le \int_0^s \int_\Omega \widetilde{\theta}(\tau)\widetilde{\chi}(\tau) d\tau.$$

By adding this inequality to (2.46) and using the Gronwall lemma, we infer that

$$\|\widetilde{\theta}\|^2_{L^2(0,s;H)} + \|\nabla(I_0\widetilde{\theta})\|^2_{L^\infty(0,s;\mathbf{H})} + \|\widetilde{\chi}\|^2_{L^\infty(0,s;H)} \le C\|\widetilde{\mathbf{p}}\|^2_{L^2(0,t;\mathbf{H})} \quad \forall s \in]0,t[, \tag{2.47}$$

for some constant $C > 0$ depending on μ, T, and ε. On the other hand, we observe that $\alpha\widetilde{\mathbf{q}}' + \widetilde{\mathbf{q}} = -\nabla\widetilde{\theta}$ a.e. in Q and $\widetilde{\mathbf{q}}(0) = \mathbf{0}$; hence we have the representation formula

$$\widetilde{\mathbf{q}}(s) = -\int_0^s \frac{1}{\alpha}\exp\left(\frac{\tau - s}{\alpha}\right)\nabla\widetilde{\theta}(\tau)d\tau \quad \forall s \in [0,T]. \tag{2.48}$$

Thus, by means of an integration by parts, we can deduce the following estimate

$$\begin{aligned}\|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}} &\le \frac{2}{\alpha^2}\|\nabla(I_0\widetilde{\theta})(s)\|^2_{\mathbf{H}} + \frac{2}{\alpha^4}\left\|\int_0^s \exp\left(\frac{\tau-s}{\alpha}\right)\nabla(I_0\widetilde{\theta})(\tau)d\tau\right\|^2_{\mathbf{H}} \\ &\le \frac{2}{\alpha^2}\|\nabla(I_0\widetilde{\theta})\|^2_{L^\infty(0,s;\mathbf{H})} + \frac{2}{\alpha^4}\|\exp(-\ \cdot\ /\alpha)\|^2_{L^1(0,s)}\|\nabla(I_0\widetilde{\theta})\|^2_{L^\infty(0,s;\mathbf{H})}.\end{aligned} \tag{2.49}$$

Since $\|\widetilde{\mathbf{p}}\|^2_{L^2(0,t;\mathbf{H})} \le \int_0^t \|\widetilde{\mathbf{p}}\|^2_{L^\infty(0,\tau;\mathbf{H})}d\tau$, by estimating the right hand side of (2.49) with the help of (2.47), it turns out that there exists a constant $C > 0$, depending on μ, T, ε, and α, such that

$$\|\widetilde{\mathbf{q}}(s)\|^2_{\mathbf{H}} \le C\int_0^t \|\widetilde{\mathbf{p}}\|^2_{L^\infty(0,\tau;\mathbf{H})}d\tau \quad \forall s \in (0,t). \tag{2.50}$$

Therefore, taking the supremum of the left hand side of (2.50) we get (2.44). □

Now we are ready to prove the existence result for the regularized problem. We need again γ to be sublinear, i.e.

$$\exists C_\gamma > 0 \quad : \quad |s| \le C_\gamma(|r| + 1) \quad \forall r \in D(\gamma),\ \forall s \in \gamma(r). \tag{2.51}$$

Proposition III.2.1. *Let $\mu > 0$ and assume that* (2.1)–(2.25) *and* (2.51) *hold.*

There exists a unique triplet $(\theta_\mu, \chi_\mu, \mathbf{q}_\mu)$ *satisfying*

$$\theta_\mu \in L^2(0,T;V) \cap L^2(0,T;V), \tag{2.52}$$
$$\theta'_{\alpha\varepsilon} \in L^2(0,T;V') + L^1(0,T;H), \tag{2.53}$$
$$\chi_\mu \in H^1(0,T;H), \tag{2.54}$$
$$\mathbf{q}_\mu \in H^1(0,T;\mathbf{H}), \tag{2.55}$$
$$\exists \xi_\mu \in L^2(Q), \quad \xi_\mu \in \beta(\chi_\mu) \quad \textit{a.e. in } Q, \tag{2.56}$$
$$(\theta_\mu + \chi_\mu)' + \mu A\theta_\mu + B\mathbf{q}_\mu = f \quad \text{ in } V', \text{ a.e. in } (0,T), \tag{2.57}$$
$$\alpha \mathbf{q}'_\mu + \mathbf{q}_\mu = -\nabla\theta_\mu + \mathbf{h}_\alpha \quad \text{ a.e. in } Q, \tag{2.58}$$
$$\varepsilon\chi'_\mu + \xi_\mu \ni \theta_\mu + \theta_D \quad \text{ a.e. in } Q, \tag{2.59}$$
$$\theta_\mu(0) = \theta_0, \ \chi_\mu(0) = \chi_0, \ \mathbf{q}_\mu(0) = \mathbf{q}_0 \quad \text{ a.e. in } Q. \tag{2.60}$$

Moreover, there exists a constant $C > 0$*, independent of* μ, α*, and* ε*, such that*

$$\|\theta_\mu\|_{L^\infty(0,T;H)} + \mu^{1/2}\|\theta_\mu\|_{L^2(0,T;V)} + \alpha^{1/2}\|\mathbf{q}_\mu\|_{L^\infty(0,T;\mathbf{H})} + \|\mathbf{q}_\mu\|_{L^2(0,T;\mathbf{H})} + \alpha\|\mathbf{q}'_\mu\|_{L^2(0,T;\mathbf{V}')} + \varepsilon^{1/2}\|\chi_\mu\|_{H^1(0,T;H)} + \|\chi_\mu\|_{L^2(0,T;H)} + \|\xi_\mu\|_{L^2(0,T;H)} \leq C. \tag{2.61}$$

Proof. It is easy to see that any fixed point $\mathbf{q}_\mu = \Sigma(\mathbf{q}_\mu)$ of the mapping Σ is such that the corresponding triplet $(\theta_\mu, \chi_\mu, \mathbf{q}_\mu)$ defined by Lemma III.2.3 yields a solution to (2.52)–(2.59), and conversely. Now, if $\mathbf{p}_1, \mathbf{p}_2 \in L^\infty(0,T;\mathbf{H})$, (2.44) implies that

$$\|\Sigma(\mathbf{p}_1) - \Sigma(\mathbf{p}_2)\|^2_{L^\infty(0,t;\mathbf{H})} \leq Ct\|\mathbf{p}_1 - \mathbf{p}_2\|^2_{L^\infty(0,t;\mathbf{H})} \quad \forall t \in [0,T]. \tag{2.62}$$

Thus, by induction it is easily seen that for any $m \in \mathbb{N}_*$

$$\|\Sigma^m(\mathbf{p}_1) - \Sigma^m(\mathbf{p}_2)\|^2_{L^\infty(0,T;\mathbf{H})} \leq \frac{(CT)^m}{m!}\|\mathbf{p}_1 - \mathbf{p}_2\|^2_{L^\infty(0,T;\mathbf{H})}. \tag{2.63}$$

This entails that for m sufficiently large, the iterated mapping Σ^m is a strict contraction and consequently Σ admits a unique fixed point, which leads to the solution we are looking for.

To get the estimate (2.61) we start by testing equation (2.57) by $\theta_\mu \in L^2(0,T;V)$ and integrating over $]0,t[$, where $t \in [0,T]$. By Hölder inequality we obtain

$$\frac{1}{2}\|\theta_\mu(t)\|^2_H + \int_0^t\int_\Omega \chi'_\mu(s)\theta_\mu(s) + \mu\|\nabla\theta_\mu\|^2_{L^2(0,t;\mathbf{H})} - \int_0^t\int_\Omega \mathbf{q}_\mu(s)\cdot\nabla\theta_\mu(s)ds$$
$$\leq \frac{1}{2}\|\theta_0\|^2_H + C\int_0^t \|\theta_\mu(s)\|^2_H ds + \int_0^t \|f(s)\|_H\|\theta_\mu(s)\|_H ds. \tag{2.64}$$

Multiplying (2.58) by $\mathbf{q}_\mu$, we deduce the inequality

$$\frac{\alpha}{2}\|\mathbf{q}_\mu(t)\|^2_{\mathbf{H}} + \frac{1}{2}\|\mathbf{q}_\mu\|^2_{L^2(0,t;\mathbf{H})} \\ \leq \frac{\alpha}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} - \int_0^t\int_\Omega \mathbf{q}_\mu(s)\cdot\nabla\theta_\mu(s)ds + \frac{1}{2}\|\mathbf{h}_\alpha\|^2_{L^2(0,t;\mathbf{H})}, \tag{2.65}$$

Now arguing as in the proof of Proposition I.5.1 we can derive the inequality

$$\varepsilon\|\chi_\varepsilon'\|^2_{L^2(0,t;H)} + C\|\chi_\varepsilon(t)\|^2_H \\ \leq C\left(1 + \int_0^t (\|\theta_\varepsilon(t)\|^2_H + \|\chi_\varepsilon(t)\|^2_H)ds\right) + \int_0^t\int_\Omega \theta_\varepsilon(s)\chi_\varepsilon'(s)ds. \tag{2.66}$$

By adding this inequality to (2.64) and (2.65), we have two cancellations and, owing to a variant of the Gronwall lemma (see Proposition (A.5.2)), we get

$$\|\theta_\mu\|^2_{L^\infty(0,t;H)} + \mu\|\nabla\theta_\mu\|^2_{L^2(0,t;\mathbf{H})} + \alpha\|\mathbf{q}_\mu\|^2_{L^\infty(0,t;\mathbf{H})} + \|\mathbf{q}_\mu\|^2_{L^2(0,t;\mathbf{H})} \\ + \varepsilon\|\chi_\mu'\|^2_{L^2(0,t;H)} + \|\chi_\mu\|^2_{L^2(0,t;H)} \leq C. \tag{2.67}$$

At this point we make a comparison of terms in (2.58) to deduce the estimate on $\alpha\mathbf{q}_\mu'$ and we argue as in (5.15) of Chapter I to get a bound of ξ_μ. Hence recalling (2.67) we get (2.61). $\square$

The estimate (2.61) is sufficient to take the limit in μ and to obtain a solution of Problem $(\mathbf{S1}_{\alpha\varepsilon})$.

Theorem III.2.1. *Assume that* (2.51) *holds. For* $\mu > 0$ *let* $(\theta_\mu, \chi_\mu, \mathbf{q}_\mu)$ *be the triplet defined by Proposition III.2.1. Then there exists a solution* $(\theta, \chi, \mathbf{q})$ *to Problem* $(\mathbf{S1}_{\alpha\varepsilon})$ *such that, at least for a subsequence of* $\mu \searrow 0$,

$$\theta_\mu \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(0,T;H), \tag{2.68}$$

$$\mu\theta_\mu \to 0 \quad \text{in } L^2(0,T;V), \tag{2.69}$$

$$\chi_\mu \rightharpoonup \chi \quad \text{in } H^1(0,T;H), \tag{2.70}$$

$$\mathbf{q}_\mu \overset{*}{\rightharpoonup} \mathbf{q} \quad \text{in } H^1(0,T;\mathbf{V}') \cap L^\infty(0,T;\mathbf{H}). \tag{2.71}$$

Proof. Let $f_{V'} \in L^2(0,T;V')$ and $f_H \in L^1(0,T;H)$ such that $f = f_{V'} + f_H$. Let us note that, on account of (2.61), $\|\theta_\mu' - f_H\|_{L^2(0,T;V')}$ turns out to be bounded independently of μ (but not of ε). Hence, there exist θ, χ, $\mathbf{q}$, nd ξ such that, possibly taking subsequences, (2.68)–(2.71),

$$\xi_\mu \rightharpoonup \xi \qquad \text{in } L^2(Q),$$

and

$$\theta'_\mu - f_H \rightharpoonup \theta' - f_H \quad \text{ in } L^2(0,T;V')$$

are satisfied as $\mu \searrow 0$. Notice also that, if we integrate (2.58) in time, by (2.61) and (2.21) we see that $\|\nabla(I_0\theta_\mu)\|_{L^\infty(0,T;\mathbf{H})}$ is uniformly bounded, whence (2.68) entails

$$I_0\theta_\mu \overset{*}{\rightharpoonup} I_0\theta \quad \text{in } W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V). \tag{2.72}$$

Then the generalized Ascoli theorem implies that

$$I_0\theta_\mu \to I_0\theta \quad \text{in } C([0,T];H). \tag{2.73}$$

In view of equations (2.57)–(2.58), conditions (2.60), and convergences (2.68)–(2.71), it is straightforward to check that θ, χ, $\mathbf{q}$ satisfy (2.26)–(2.28), (2.30)–(2.32), (2.34), and

$$\varepsilon\chi' + \xi = \theta + \theta_D \quad \text{ in } Q.$$

Therefore, to verify that the triplet $(\theta, \chi, \mathbf{q})$ solves Problem $(\mathbf{S1}_{\alpha\varepsilon})$, it remains to show (2.33). It suffices to show that

$$\limsup_{\mu\searrow 0} \int_Q \xi_\mu\chi_\mu \le \int_Q \xi\chi.$$

Now, integrating by parts leads to

$$\begin{aligned}
\int_Q \xi_\mu\chi_\mu &= \int_0^T\int_\Omega (\theta_\mu(t) + \theta_D(t) - \varepsilon\chi'_\mu(t))\chi_\mu(t)dt \\
&= \int_\Omega (I_0\theta_\mu)(T)\chi_\mu(T) - \int_0^T\int_\Omega (I_0\theta_\mu)(t)\chi'_\mu(t)dt \\
&\quad + \int_0^T\int_\Omega \theta_D(t)\chi_\mu(t)dt - \frac{\varepsilon}{2}\left(\|\chi_\mu(T)\|_H^2 - \|\chi_0\|_H^2\right)
\end{aligned} \tag{2.74}$$

and we can take the upper limit in (2.74) as $\mu \searrow 0$. Thanks to (2.73), (2.70), and to the weak lower semicontinuity of the norms, thus we get

$$\begin{aligned}
\limsup_{\mu\searrow 0} \int_Q \xi_\mu\chi_\mu &\le \int_0^T\int_\Omega \theta_D(t)\chi(t)dt - \frac{\varepsilon}{2}\left(\|\chi(T)\|_H^2 - \|\chi_0\|_H^2\right) \\
&= \int_0^T\int_\Omega (\theta(t) + \theta_D(t) - \varepsilon\chi'(t))\chi(t)dt = \int_Q \xi\chi
\end{aligned} \tag{2.75}$$

for θ and χ, which proves (2.33). □

Remark III.2.2. We remark that we are not able to prove any uniqueness result about the solutions of Problem $(\mathbf{S1}_{\alpha\epsilon})$, and in fact this is still an open question.

III.3 Convergence to the Stefan problem

In this section we investigate the asymptotic behaviour of the solutions of Problem $(\mathbf{S1}_{\alpha\epsilon})$ as α and ε go to zero. We will show that in a suitable functional space their limit is in fact the solution of the Stefan problem.

Now we restate the existence and uniqueness result about the Stefan problem adapting it to our new situation.

Problem $(\mathbf{S}')$**.** Assume that (2.1)–(2.25) hold. Find a pair (θ, χ) such that

$$\theta \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{3.1}$$
$$(\theta+\chi)' \in L^2(0,T;V') \tag{3.2}$$
$$\chi \in L^2(Q), \tag{3.3}$$
$$(\theta+\chi)' + A\theta = f - B\mathbf{h} \quad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{3.4}$$
$$\chi \in \gamma(\theta+\theta_D) \quad \text{a.e. in } Q, \tag{3.5}$$
$$(\theta+\chi)(0) = \theta_0 + \chi_0 \quad \text{in } V'. \tag{3.6}$$

♠

Rephrasing Theorem I.5.1 we can state the following

Theorem III.3.1. *Assume* (2.51)*. Then there exists a unique solution to Problem* $(\mathbf{S}')$*.*

In order to prove that the solutions of Problem $(\mathbf{S1}_{\alpha\epsilon})$ converge to the solution of the Stefan Problem $(\mathbf{S}')$, we first need to recover some a priori estimates. Let us point out that we cannot simply pass to the lower limit in (2.61) as $\mu \searrow 0$. In fact, the resulting estimate would hold only for the solutions of Problem $(\mathbf{S1}_{\alpha\varepsilon})$ which are obtained as limit, in the sense of (2.68)–(2.71), of some subsequence of $(\theta_\mu, \chi_\mu, \mathbf{q}_\mu)$. Now we are going to prove that the validity of (2.61) extends to any solution of Problem $(\mathbf{S1}_{\alpha\varepsilon})$. We need to strenghten the assumption on f, we will suppose that

$$f \in L^2(0,T;V') \cap L^1(0,T;H). \tag{3.7}$$

Of course, it is not restrictive to assume α and ε to be less than 1. The next uniqueness lemma is the key to get (2.61).

Lemma III.3.1. *Assume that* $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ *is a solution to Problem* $(\mathbf{P}_{\alpha\varepsilon})$. *Then there exists a unique pair* $(u_{\alpha\varepsilon}, \mathbf{p}_{\alpha\varepsilon})$ *such that*

$$u_{\alpha\varepsilon} \in H^1(0,T;V') \cap L^\infty(0,T;H), \tag{3.8}$$

$$\mathbf{p}_{\alpha\varepsilon} \in H^1(0,T;\mathbf{V}') \cap L^\infty(0,T;\mathbf{H}), \tag{3.9}$$

$$u'_{\alpha\varepsilon} + B\mathbf{p}_{\alpha\varepsilon} = f - \chi'_{\alpha\varepsilon} \quad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{3.10}$$

$$\alpha\mathbf{p}'_{\alpha\varepsilon} + \mathbf{p}_{\alpha\varepsilon} = \mathbf{L}u_{\alpha\varepsilon} + \mathbf{h}_\alpha \quad \text{in } \mathbf{V}', \quad \text{a.e. in }]0,T[, \tag{3.11}$$

$$u_{\alpha\varepsilon}(0) = \theta_0 \quad \text{in } V', \qquad \mathbf{p}_{\alpha\varepsilon}(0) = \mathbf{q}_0 \quad \text{in } \mathbf{V}'. \tag{3.12}$$

In particular, it turns out that $(u_{\alpha\varepsilon}, \mathbf{p}_{\alpha\varepsilon})$ *coincides with* $(\theta_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$.

Proof. Concerning the existence, it is enough to observe that $(\theta_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ satisfies (3.8)–(3.12). Now, for convenience we omit the subscript $\alpha\varepsilon$. Let us assume that $(u_i, \mathbf{p}_i)$, $i = 1, 2$, are two solutions and set $\widetilde{u} := u_1 - u_2$, $\widetilde{\mathbf{p}} := \mathbf{p}_1 - \mathbf{p}_2$. Then, integrating (3.10)–(3.11) in time, with the help of (3.12) we get

$$\widetilde{u} + B(I_0\widetilde{\mathbf{p}}) = 0 \quad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{3.13}$$

$$\alpha\widetilde{\mathbf{p}} + I_0\widetilde{\mathbf{p}} = \mathbf{L}(I_0\widetilde{u}) \quad \text{in } \mathbf{V}', \text{ a.e. in }]0,T[. \tag{3.14}$$

Since (3.8)–(3.9) hold, by comparison in (3.13) and in (3.13) we infer that $B(I_0\widetilde{\mathbf{p}}) \in L^\infty(0,T;H)$ and $\mathbf{L}(I_0\widetilde{u}) \in L^\infty(0,T;\mathbf{H})$. Therefore, Lemma III.2.1 and Lemma III.2.2 yield respectively $I_0\widetilde{\mathbf{p}} \in L^\infty(0,T;\mathbf{V})$ and $I_0\widetilde{u} \in L^\infty(0,T;V)$, which allow us to test (3.13) by $I_0\widetilde{u}$ and (3.14) by $I_0\widetilde{\mathbf{p}}$. Integrating and adding the resulting equalities, we easily obtain

$$\frac{1}{2}\|(I_0\widetilde{u})(t)\|_H^2 + \frac{\alpha}{2}\|(I_0\widetilde{\mathbf{p}})(t)\|_{\mathbf{H}}^2 + \|I_0\widetilde{\mathbf{p}}\|_{L^2(0,t;\mathbf{H})}^2 = 0 \qquad \forall t \in [0,T]. \tag{3.15}$$

Hence we infer that $\widetilde{u} = 0$ and $\widetilde{\mathbf{p}} = \mathbf{0}$. □

Now, we use the same regularization procedure as in Section III.2.

Lemma III.3.2. *Assume that and* (2.51) *and* (3.7) *hold. Moreover suppose that* $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ *is a solution to Problem* $(\mathbf{S1}_{\alpha\varepsilon})$ *and let* $\mu \in (0,1)$. *Then, there exists a unique pair* $(u_\mu, \mathbf{p}_\mu)$ *such that*

$$u_\mu \in H^1(0,T;V') \cap L^2(0,T;V), \tag{3.16}$$

$$\mathbf{p}_\mu \in H^1(0,T;\mathbf{H}), \tag{3.17}$$

$$u'_\mu + \mu A u_\mu + B\mathbf{p}_\mu = f - \chi'_{\alpha\varepsilon} \quad \text{in } V', \text{ a.e. in } (0,T), \tag{3.18}$$

$$\alpha\mathbf{p}'_\mu + \mathbf{p}_\mu = -\nabla u_\mu + \mathbf{h}_\alpha \quad \text{a.e. in } Q, \tag{3.19}$$

$$u_\mu(0) = \theta_0, \quad \mathbf{p}_\mu(0) = \mathbf{q}_0 \quad \text{a.e. in } Q. \tag{3.20}$$

Moreover there exists a constant $C > 0$, independent of μ, such that

$$\|u_\mu\|_{H^1(0,T;V')\cap L^\infty(0,T;H)} + \mu^{1/2}\|u_\mu\|_{L^2(0,T;V)} \\ + \alpha^{1/2}\|\mathbf{p}_\mu\|_{L^\infty(0,T;\mathbf{H})} + \|\mathbf{p}_\mu\|_{L^2(0,T;\mathbf{H})} + \alpha\|\mathbf{p}'_\mu\|_{L^2(0,T;\mathbf{V}')} \le C. \tag{3.21}$$

Proof. To show existence and uniqueness of $(u_\mu, \mathbf{p}_\mu)$, one can argue, for instance, as in the previous section. Regarding estimate (3.21), we first test (3.18) by u_μ and (3.19) by $\mathbf{p}_\mu$. Take the sum and integrate over $]0,t[$. Thanks to the Hölder inequality, we obtain

$$\frac{1}{2}\|u_\mu(t)\|_H^2 + \mu\|\nabla u_\mu\|^2_{L^2(0,t;\mathbf{H})} + \frac{\alpha}{2}\|\mathbf{p}_\mu(t)\|^2_{\mathbf{H}} + \frac{1}{2}\|\mathbf{p}_\mu\|^2_{L^2(0,t;\mathbf{H})} \\ \le \frac{1}{2}\|\theta_0\|_H^2 + \frac{\alpha}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} + \frac{1}{2}\|\mathbf{h}_\alpha\|^2_{L^2(0,t;\mathbf{H})} + \int_0^t\int_\Omega (f(s) - \chi'(s))u_\mu(s)ds. \tag{3.22}$$

Since $\int_0^t\int_\Omega (f(s)-\chi'(s))u_\mu(s)ds \le \int_0^t \|f(s)-\chi'(s)\|_H\|u_\mu(s)\|_H ds$, an application of the Gronwall lemma enables us to infer that

$$\|u_\mu\|^2_{L^\infty(0,T;H)} + \mu\|\nabla u_\mu\|^2_{L^2(0,T;\mathbf{H})} + \alpha\|\mathbf{p}_\mu\|^2_{L^\infty(0,T;\mathbf{H})} + \|\mathbf{p}_\mu\|^2_{L^2(0,T;\mathbf{H})} \le C, \tag{3.23}$$

for some constant $C > 0$ which is independent of μ. Now, a comparison in (3.18) and (3.19) allows us to get (3.21). □

Lemma III.3.1 and III.3.2 allow us to deduce the estimate (2.61) for any solution of Problem $(\mathbf{S1}_{\alpha}\varepsilon)$, as shown by the following proposition.

Proposition III.3.1. *Assume that and* (2.51) *and* (3.7) *hold. Then there exists a constant $C > 0$, independent of α and ε, such that for all solutions $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ of $(\mathbf{S1}_{\alpha\varepsilon})$ there holds*

$$\|\theta_{\alpha\varepsilon}\|_{L^\infty(0,T;H)} + \|\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon}\|_{H^1(0,T;V')} + \alpha^{1/2}\|\mathbf{q}_{\alpha\varepsilon}\|_{L^\infty(0,T;\mathbf{H})} \\ + \|\mathbf{q}_{\alpha\varepsilon}\|_{L^2(0,T;\mathbf{H})} + \alpha\|\mathbf{q}'_{\alpha\varepsilon}\|_{L^2(0,T;\mathbf{V}')} + \varepsilon^{1/2}\|\chi_{\alpha\varepsilon}\|_{H^1(0,T;H)} + \|\chi_{\alpha\varepsilon}\|_{L^\infty(Q)} \le C. \tag{3.24}$$

Proof. Let us omit the subscript $\alpha\varepsilon$. Let the triplet $(\theta, \chi, \mathbf{q})$ solve Problem $(\mathbf{S1}_{\alpha\varepsilon})$. Then, by Corollary I.3.1 we have that

$$\varepsilon\|\chi'\|^2_{L^2(0,t;H)} \le C + \int_0^t\int_\Omega \chi'(s)\theta(s)ds. \tag{3.25}$$

for some constant $C > 0$, independent of α, ε, and μ. On the other hand, thanks to Lemma III.3.2 we can pass to the limit on u_μ and $\mathbf{p}_\mu$, by weak or weak star compactness. Recalling Theorem III.2.1, it is clear that the limit pair $(u, \mathbf{p})$ solves problem (3.8)–(3.12). Hence, by virtue of Lemma III.3.1, we have that

$$u_\mu \overset{*}{\rightharpoonup} \theta \quad \text{in } H^1(0,T;V') \cap L^\infty(0,T;H), \tag{3.26}$$

$$\mu u_\mu \to 0 \quad \text{in } L^2(0,T;V), \tag{3.27}$$

$$\mathbf{p}_\mu \overset{*}{\rightharpoonup} \mathbf{q} \quad \text{in } H^1(0,T;\mathbf{V}') \cap L^\infty(0,T;\mathbf{H}) \tag{3.28}$$

as $\mu \searrow 0$, and the convergences hold for the entire sequences. Now we argue as in the proof of Theorem II.3.1, formulae (3.18) and (3.19). Note that (3.28) entails

$$\mathbf{p}_\mu(t) \rightharpoonup \mathbf{q}(t) \quad \text{in } \mathbf{V}', \quad \forall t \in [0,T]. \tag{3.29}$$

In view of (3.17) and (2.31), we point out that $\mathbf{p}_\mu \in C([0,T];\mathbf{H})$ for all $0 < \mu < 1$ and $\mathbf{q}$ is weakly continuous from $[0,T]$ to $\mathbf{H}$, so it makes sense to consider $\mathbf{q}(t) \in \mathbf{H}$, for any $t \in [0,T]$. As from (3.21) $\|\mathbf{p}_\mu\|_{L^\infty(0,T;\mathbf{H})}$ is uniformly bounded with respect to μ, it follows that for any $t \in [0,T]$ there exists a subsequence of $\mathbf{p}_\mu(t)$ that weakly converges to some function $\mathbf{r}_t$ in $\mathbf{H}$. Therefore, owing to (3.29) and the uniqueness of the limit we get $\mathbf{r}_t = \mathbf{q}(t) \in \mathbf{H}$ and

$$\mathbf{p}_\mu(t) \rightharpoonup \mathbf{q}(t) \quad \text{in } \mathbf{H} \tag{3.30}$$

as $\mu \searrow 0$, and this convergence holds for the entire sequence and for all $t \in [0,T]$. Exactly in the same way we see that

$$u_\mu(t) \rightharpoonup \theta(t) \quad \text{in } H, \quad \forall t \in [0,T]. \tag{3.31}$$

Now, owing to the convergences (3.29)–(3.30), (3.28), and (3.26), from (3.22) we infer

$$\begin{aligned}
&\frac{1}{2}\|\theta(t)\|_H^2 + \frac{\alpha}{2}\|\mathbf{q}(t)\|_{\mathbf{H}} + \frac{1}{2}\|\mathbf{q}\|^2_{L^2(0,t;\mathbf{H})} \\
&\le \liminf_{\mu\searrow 0} \frac{1}{2}\|u_\mu(t)\|_H^2 + \liminf_{\mu\searrow 0} \frac{\alpha}{2}\|\mathbf{p}_\mu(t)\|^2_{\mathbf{H}} + \liminf_{\mu\searrow 0} \frac{1}{2}\|\mathbf{p}_\mu\|^2_{L^2(0,t;\mathbf{H})} \\
&\le \liminf_{\mu\searrow 0} \left(\frac{1}{2}\|\theta_0\|_H^2 + \frac{\alpha}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} + \frac{1}{2}\|\mathbf{h}_\alpha\|^2_{L^2(0,t;\mathbf{H})} + \int_0^t\int_\Omega (f(s) - \chi'(s))u_\mu(s)ds \right) \\
&= \frac{1}{2}\|\theta_0\|_H^2 + \frac{\alpha}{2}\|\mathbf{q}_0\|^2_{\mathbf{H}} + \frac{1}{2}\|\mathbf{h}_\alpha\|^2_{L^2(0,t;\mathbf{H})} + \int_0^t\int_\Omega (f(s) - \chi'(s))\theta(s)ds
\end{aligned} \tag{3.32}$$

for all $t \in [0,T]$. Then, adding (3.25) and (3.32), we obtain

$$\frac{1}{2}\|\theta(t)\|_H^2 + \frac{\alpha}{2}\|\mathbf{q}(t)\|_{\mathbf{H}} + \frac{1}{2}\|\mathbf{q}\|^2_{L^2(0,t;H)} + \varepsilon\|\chi'\|^2_{L^2(0,t;H)} \le C + \int_0^t\int_\Omega f(s)\theta(s)ds \tag{3.33}$$

for all $t \in [0,T]$ and for some constant $C > 0$, which does not depend on α and ε. Finally, an application of Gronwall Lemma allows us to conclude the proof. □

Using the estimate established in Proposition III.3.1 we can finally start the limit procedure and obtain the desired convergence. The following technical lemma will be useful.

Lemma III.3.3. *Let* $\mathbf{u} \in H^1(0,T;\mathbf{V}') \cap L^\infty(0,T;\mathbf{H})$ *and* $\mathbf{v} \in H^1(0,T;\mathbf{H}) \cap L^\infty(0,T;\mathbf{V})$. *Then the function* $t \mapsto {}_{\mathbf{V}'}\langle \mathbf{u}(t), \mathbf{v}(t)\rangle_{\mathbf{V}}$ *belongs to* $H^1(0,T)$ *and the formula*

$$ {}_{\mathbf{V}'}\langle \mathbf{u}(t), \mathbf{v}(t)\rangle_{\mathbf{V}} = {}_{\mathbf{V}'}\langle \mathbf{u}(0), \mathbf{v}(0)\rangle_{\mathbf{V}} + \int_0^t \left\{ {}_{\mathbf{V}'}\langle \mathbf{u}'(s), \mathbf{v}(s)\rangle_{\mathbf{V}} + (\mathbf{u}(s), \mathbf{v}'(s))_{\mathbf{H}} \right\} ds \tag{3.34} $$

holds for any $t \in [0,T]$. *In particular, note that* $\mathbf{v}$ *is weakly continuous from* $[0,T]$ *to* $\mathbf{V}$; *moreover, since* $\mathbf{u}$ *is weakly continuous from* $[0,T]$ *to* $\mathbf{H}$, *the first two duality pairings in* (3.34) *can be read as inner products in* $\mathbf{H}$.

Proof. First of all, we extend $\mathbf{u}$ to $]-\infty, 0[$ by the constant value $\mathbf{u}(0)$ and introduce

$$ \mathbf{u}_n(t) := \frac{1}{1/n} \int_{t-1/n}^t \mathbf{u}(s) ds, \quad t \in [0,T], \ n \in \mathbb{N}, \tag{3.35} $$

this approximating functions belonging to $H^1(0,T;\mathbf{H})$. Then it is known that the formula (3.34) applies to $\mathbf{u}_n$ and $\mathbf{v}$, i.e.,

$$ \begin{aligned} & {}_{\mathbf{V}'}\langle \mathbf{u}_n(t), \mathbf{v}(t)\rangle_{\mathbf{V}} - {}_{\mathbf{V}'}\langle \mathbf{u}_n(0), \mathbf{v}(0)\rangle_{\mathbf{V}} \\ & = \int_0^t \left\{ {}_{\mathbf{V}'}\langle \mathbf{u}_n'(s), \mathbf{v}(s)\rangle_{\mathbf{V}} + (\mathbf{v}'(s), \mathbf{u}_n(s))_{\mathbf{H}} \right\} ds \end{aligned} \tag{3.36} $$

for all $t \in [0,T]$. Now, note that for any $t \in [0,T]$

$$ \mathbf{u}_n(t) \to \mathbf{u}(t) \quad \text{in } \mathbf{V}', \tag{3.37} $$
$$ \mathbf{u}_n(t) \rightharpoonup \mathbf{u}(t) \quad \text{in } \mathbf{H}, \tag{3.38} $$

by the continuity of $\mathbf{u}$ from $[0,T]$ to $\mathbf{V}'$ and by the boundedness of $\|\mathbf{u}_n\|_{L^\infty(0,T;\mathbf{H})}$ independently of n. Moreover, we have that

$$ \mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{H}) \tag{3.39} $$

(actually it can be shown that this convergence is strong), and

$$ \mathbf{u}_n' \to \mathbf{u}' \quad \text{in } L^2(0,T;\mathbf{V}') \tag{3.40} $$

(see, e.g., [5, Prop. A.6, p. 154]). Therefore, passing to the limit in (3.36) as $n \to \infty$, we obtain (3.34) and the lemma is completely proved. □

Remark III.3.1. As we observed before, the convergence (3.39) is actually strong. But one cannot prove this without the basic rudiments of measure theory. Since the set of Lebesgue points has negligible complement, for almost all $t \in]0,T[$ we have

$$\lim_{n\to\infty} \|\mathbf{u}_n(t) - \mathbf{u}\|_{\mathbf{H}}^2 = \lim_{n\to\infty} \left\| \int_{t-1/n}^{t} \mathbf{u}(s)ds - \mathbf{u}(t) \right\|_{\mathbf{H}}^2 = 0,$$

moreover

$$\begin{aligned}
\|\mathbf{u}_n(t) - \mathbf{u}\|_{\mathbf{H}}^2 &= \left\| \int_{t-1/n}^{t} \mathbf{u}(s)ds - \mathbf{u}(t) \right\|_{\mathbf{H}}^2 \\
&= \left\| \frac{1}{1/n} \int_{t-1/n}^{t} (\mathbf{u}(s) - \mathbf{u}(t))ds \right\|_{\mathbf{H}}^2 \\
&\leq \frac{1}{1/n} \int_{t-1/n}^{t} \|\mathbf{u}(s) - \mathbf{u}(t)\|_{\mathbf{H}}^2 \\
&\leq 4\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{H})}^2.
\end{aligned}$$

The Lebesgue dominated convergence theorem allows to conclude.

Here is the main theorem of this section.

Theorem III.3.2. *For any $\alpha, \varepsilon > 0$, let $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ be an arbitrary solution of Problem* $(\mathbf{S1}_{\alpha\varepsilon})$*. Moreover let (θ, χ) be the unique solution to Problem* $(\mathbf{S'})$*.*

$$\theta_{\alpha\varepsilon} \overset{*}{\rightharpoonup} \theta \quad \textit{in } L^\infty(0,T;H), \tag{3.41}$$

$$\chi_{\alpha\varepsilon} \rightharpoonup \chi \quad \textit{in } L^2(0,T;H), \tag{3.42}$$

as $\alpha, \varepsilon \searrow 0$.

Proof. By virtue (3.24) of there exists a triplet $(\theta, \chi, \mathbf{q})$ such that, possibly taking subsequences,

$$\theta_{\alpha\varepsilon} \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(0,T;H), \tag{3.43}$$

$$\chi_{\alpha\varepsilon} \overset{*}{\rightharpoonup} \chi \quad \text{in } L^2(0,T;H), \tag{3.44}$$

$$\varepsilon\chi'_{\alpha\varepsilon} \to 0 \quad \text{in } L^2(0,T;H), \tag{3.45}$$

$$\mathbf{q}_{\alpha\varepsilon} \rightharpoonup \mathbf{q} \quad \text{in } L^2(0,T;\mathbf{H}), \tag{3.46}$$

$$\alpha\mathbf{q}_{\alpha\varepsilon} \to \mathbf{0} \quad \text{in } L^\infty(0,T;\mathbf{H}), \tag{3.47}$$

$$\alpha\mathbf{q}_{\alpha\varepsilon} \rightharpoonup \mathbf{0} \quad \text{in } H^1(0,T;\mathbf{V}'), \tag{3.48}$$

$$\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon} \rightharpoonup \theta + \chi \quad \text{in } H^1(0,T;V') \tag{3.49}$$

as $\alpha, \varepsilon \searrow 0$. Therefore we have that

$$(\theta + \chi)' + B\mathbf{q} = f \quad \text{in } V', \text{ a.e. in }]0,T[, \tag{3.50}$$

$$\mathbf{q} = \mathbf{L}\theta + \mathbf{h} \quad \text{in } \mathbf{V}', \text{ a.e. in }]0,T[. \tag{3.51}$$

Now, from (3.46), (3.51), and Lemma III.2.2 it turns out that $\theta \in L^2(0,T;V)$ and $\mathbf{q} = -\nabla\theta + \mathbf{h}$ a.e. in Q. Then it is straightforward to deduce (3.4) from (3.50) and (3.51). It remains to prove the nonlinear relation (3.5). To this aim, let us note preliminarily that integrating equation (2.31) yields

$$B(I_0\mathbf{q}_{\alpha\varepsilon}) = I_0 f + \theta_0 + \chi_0 - \theta_{\alpha\varepsilon} - \chi_{\alpha\varepsilon} =: z_{\alpha\varepsilon} \tag{3.52}$$

and $z_{\alpha\varepsilon}$ is uniformly bounded in $L^\infty(0,T;H)$ because of (3.24). This means that for almost all $t \in]0,T[$

$$-\int_\Omega (I_0\mathbf{q}_{\alpha\varepsilon})(t) \cdot \nabla v = \int_\Omega z_{\alpha\varepsilon}(t) v \qquad \forall v \in V, \tag{3.53}$$

and, according to Lemma III.2.1 $\|I_0\mathbf{q}_{\alpha\varepsilon}\|_{L^\infty(0,T;\mathbf{V})}$ is bounded independently of α, ε. Hence we have that

$$I_0\mathbf{q}_{\alpha\varepsilon} \stackrel{*}{\rightharpoonup} I_0\mathbf{q} \quad \text{in } L^\infty(0,T;\mathbf{V}). \tag{3.54}$$

Now, to prove (3.5) observe that taking the limit in equation (2.33) we find that

$$\xi = \theta + \theta_D \qquad \text{a.e. in } Q. \tag{3.55}$$

Therefore we have to show that

$$\limsup_{\alpha,\varepsilon \searrow 0} \int_Q \xi_{\alpha\varepsilon}\chi_{\alpha\varepsilon} \le \int_Q \xi\chi. \tag{3.56}$$

We can write

$$\int_Q \xi_{\alpha\varepsilon}\chi_{\alpha\varepsilon} = \int_Q (\theta_{\alpha\varepsilon} + \theta_D - \varepsilon\chi'_{\alpha\varepsilon})\chi_{\alpha\varepsilon}. \tag{3.57}$$

By exploiting (3.52)–(3.53), as $z_{\alpha\varepsilon} = \operatorname{div}(I_0\mathbf{q}_{\alpha\varepsilon})$ it is easy to derive

$$\int_Q \theta_{\alpha\varepsilon}\chi_{\alpha\varepsilon} = \int_Q \theta_{\alpha\varepsilon}(I_0 f + \theta_0 + \chi_0) - \|\theta_{\alpha\varepsilon}\|^2_{L^2(Q)} - \int_Q \theta_{\alpha\varepsilon} \operatorname{div}(I_0\mathbf{q}_{\alpha\varepsilon}). \tag{3.58}$$

On the other hand, if we test (2.32) by $I_0\mathbf{q}_{\alpha\varepsilon} \in L^\infty(0,T;\mathbf{V})$, we get

$$\begin{aligned}
\int_Q \theta_{\alpha\varepsilon} \operatorname{div}(I_0\mathbf{q}_{\alpha\varepsilon}) &= \int_0^T {}_{\mathbf{V}'}\langle \alpha\mathbf{q}'_{\alpha\varepsilon}(s), (I_0\mathbf{q}_{\alpha\varepsilon})(s)\rangle_{\mathbf{V}} ds \\
&\quad + \int_0^T {}_{\mathbf{V}'}\langle \mathbf{q}_{\alpha\varepsilon}(s), (I_0\mathbf{q}_{\alpha\varepsilon})(s)\rangle_{\mathbf{V}} ds - \int_Q \mathbf{h}_\alpha \cdot I_0\mathbf{q}_{\alpha\varepsilon} \\
&= \int_0^T {}_{\mathbf{V}'}\langle \alpha\mathbf{q}'_{\alpha\varepsilon}(s), (I_0\mathbf{q}_{\alpha\varepsilon})(s)\rangle_{\mathbf{V}} ds \\
&\quad + \frac{1}{2}\|(I_0\mathbf{q}_{\alpha\varepsilon})(T)\|^2_{\mathbf{H}} - \int_Q \mathbf{h}_\alpha \cdot I_0\mathbf{q}_{\alpha\varepsilon}. \qquad (3.59)
\end{aligned}$$

Since $\widehat{\mathbf{q}}_{\alpha\varepsilon} \in H^2(0,T;\mathbf{V}') \cap W^{1,\infty}(0,T;\mathbf{H}) \cap L^\infty(0,T;\mathbf{V})$, by Lemma III.3.3 the function $t \mapsto {}_{\mathbf{V}'}\langle \alpha\mathbf{q}_{\alpha\varepsilon}(t), (I_0\mathbf{q}_{\alpha\varepsilon})(t)\rangle_{\mathbf{V}}$ is absolutely continuous and

$$\begin{aligned}
&\int_0^T {}_{\mathbf{V}'}\langle \alpha\mathbf{q}'_{\alpha\varepsilon}(s), (I_0\mathbf{q}_{\alpha\varepsilon})(s)\rangle_{\mathbf{V}} ds \\
&= {}_{\mathbf{V}'}\langle \alpha\mathbf{q}_{\alpha\varepsilon}(T), (I_0\mathbf{q}_{\alpha\varepsilon})(T)\rangle_{\mathbf{V}} - \int_0^T (\alpha\mathbf{q}_{\alpha\varepsilon}(s), \mathbf{q}_{\alpha\varepsilon}(s))_{\mathbf{H}} ds. \qquad (3.60)
\end{aligned}$$

Note that this quantity goes to 0 as $\alpha, \varepsilon \searrow 0$, because of (3.46)–(3.47) and (3.54). Hence, using (3.52), (2.21), and the weak lower semicontinuity of the norms, we infer that

$$\liminf_{\alpha,\varepsilon\searrow 0} \int_Q \theta_{\alpha\varepsilon} \operatorname{div}(I_0\mathbf{q}_{\alpha\varepsilon}) \geq \frac{1}{2}\|(I_0\mathbf{q})(T)\|^2_{\mathbf{H}} - \int_Q \mathbf{h} \cdot I_0\mathbf{q}. \qquad (3.61)$$

At this point, we can recall (3.51) and (3.50) to point out that

$$\begin{aligned}
\frac{1}{2}\|(I_0\mathbf{q})(T)\|^2_{\mathbf{H}} - \int_Q \mathbf{h} \cdot I_0\mathbf{q} &= \int_0^T {}_{\mathbf{V}'}\langle \mathbf{q}(s), (I_0\mathbf{q})(s)\rangle_{\mathbf{V}} - \int_Q \mathbf{h} \cdot I_0\mathbf{q} \\
&= \int_Q \theta \operatorname{div}(I_0\mathbf{q}) = -\int_Q \nabla\theta \cdot I_0\mathbf{q} \\
&= \int_0^T {}_{\mathbf{V}'}\langle B(I_0\mathbf{q})(s), \theta(s)\rangle_{\mathbf{V}} ds \\
&= \int_Q (I_0 f + \theta_0 + \chi_0 - \theta - \chi)\theta. \qquad (3.62)
\end{aligned}$$

Thus, taking the upper limit in (3.58), on account of (3.43) and (3.61)–(3.62) we deduce that

$$\begin{aligned}
\limsup_{\alpha,\varepsilon\searrow 0} \int_Q \theta_{\alpha\varepsilon}\chi_{\alpha\varepsilon} &\leq \int_Q \theta(I_0 f + \theta_0 + \chi_0) - \|\theta\|^2_{L^2(Q)} \\
&\quad - \frac{1}{2}\|(I_0\mathbf{q})(T)\|^2_{\mathbf{H}} + \int_Q \mathbf{h} \cdot I_0\mathbf{q} = \int_Q \theta\chi. \qquad (3.63)
\end{aligned}$$

Now, thanks to (3.44)–(3.45), we have that

$$\lim_{\alpha,\varepsilon\searrow 0}\int_Q \varepsilon\chi'_{\alpha\varepsilon}\chi_{\alpha\varepsilon}=0. \tag{3.64}$$

Therefore collecting (3.63)–(3.64) we deduce (3.56) and (3.5) is proved. Finally, let us note that the uniqueness of the solution to Problem **(S′)** implies that the whole family $(\theta_{\alpha\varepsilon},\chi_{\alpha\varepsilon})$ converges as α and ε tend to 0. $\square$

Appendix A

Some analysis tools

A.1 Sobolev spaces

In this section we recall the main definitions and properties about Sobolev spaces. We refer the reader to the monographs [1], [21], [24], and [39]. The Bochner integral for Banach-valued functions is needed, see [22] and [17]. Let Ω be an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}_*$) and let E be a real Banach space. An element $\alpha = (\alpha_1, \dots, \alpha_d)$ of $\mathbb{N}^d$ is called *multi-index*, and the *lenght* of α is defined as the integer $|\alpha| := \sum_{i=1}^d \alpha_i$. If $u \in C^\infty(\Omega)$ then $D^\alpha u := D_1^{\alpha_1} \cdots D_d^{\alpha_d} u$. We denote by $C_c^\infty(\Omega)$ the space of infinitely differentiable functions from Ω in $\mathbb{R}$, having compact support in Ω.

Definition A.1.1. A *distribution* on Ω (with values in E) is a linear map $T : C_c^\infty(\Omega) \longrightarrow E$ such that the following condition holds:

$$\left.\begin{array}{c} (\varphi_n) \text{ sequence in } C_c^\infty(\Omega) \\ D^\alpha \varphi_n \to D^\alpha \varphi \text{ uniformly on } \Omega \quad \forall \alpha \in \mathbb{N}^d \\ \exists K \text{ compact in } \Omega \ : \ \mathrm{supp} \varphi_n \subseteq K \ \forall n \in \mathbb{N} \end{array}\right\} \Longrightarrow T(\varphi_n) \to T(\varphi).$$

The set of all distributions on Ω with values in E is denoted by $\mathscr{D}'(\Omega; E)$ or simply by $\mathscr{D}'(\Omega)$ if $E = \mathbb{R}$. $\diamondsuit$

Now we give the definition of derivative of a distribution.

Definition A.1.2. Let $T \in \mathscr{D}'(\Omega; E)$. The *(distributional) derivative with respect to the i-th coordinate* of T is denoted by $D_i T$ and is defined by

$$D_i T(\varphi) := -T(D_i \varphi), \qquad \varphi \in C_c^\infty(\Omega).$$

$\diamondsuit$

The derivative of a distribution has very good properties, for instance we have that

$$D_i D_j T = D_j D_i T \qquad \forall i, j \in \mathbb{N}, \quad \forall T \in \mathscr{D}'(\Omega; E).$$

Higher order derivatives are defined in the expected way:

$$D^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi), \qquad \varphi \in C_c^\infty(\Omega)$$

for all multi-index α.

The space $L^1_{\mathrm{loc}}(\Omega; E)$ can be identified with a subspace of $\mathscr{D}'(\Omega; E)$ by means of the linear inijective application which to any $u \in L^1_{\mathrm{loc}}(\Omega; E)$ assignes the distribution T_u defined by

$$T_u(\varphi) := \int_\Omega \varphi u, \qquad \varphi \in C_c^\infty(\Omega).$$

In this sense we can write the inclusion $L^1_{\mathrm{loc}}(\Omega; E) \subseteq \mathscr{D}'(\Omega; E)$.

In the space of distributions a notion of convergence can be defined .

Definition A.1.3. Let $(T_n)_{n \in \mathbb{N}_*}$ be a sequence in $\mathscr{D}'(\Omega; E)$. We say that T_n *converges in the sense of distributions* to the distribution T, and we write $T_n \to T$ in $\mathscr{D}'(\Omega; E)$, if

$$T_n(\varphi) \to T(\varphi) \qquad \forall \varphi \in C_c^\infty(\Omega).$$

♢

The following property shows the good behaviour of the distributional convergence.

Proposition A.1.1. *Let $(T_n)_{n \in \mathbb{N}_*}$ be a sequence in $\mathscr{D}'(\Omega; E)$ such that $T_n \to T$ in $\mathscr{D}'(\Omega; E)$ to some distribution T. Then*

$$T_n \to T \text{ in } \mathscr{D}'(\Omega; E) \qquad \Longrightarrow \qquad D^\alpha T_n \;\to\; D^\alpha T \text{ in } \mathscr{D}'(\Omega; E)$$

for all multi-index α.

In the sequel the notation D^α will denote the distributional derivative.

Definition A.1.4. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega; E)$ is defined by

$$W^{k,p}(\Omega; E) := \{u \in L^p(\Omega; E) \;:\; D^\alpha u \in L^p(\Omega; E) \; \forall \alpha \in \mathbb{N}^d, \; |\alpha| \le k\}.$$

If $u \in W^{k,p}(\Omega; E)$, we set

$$\|u\|_{W^{k,p}(\Omega;E)} := \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega;E)} \right)^{1/p}$$

whenever $p \in [1, \infty[$, and we define $\|u\|_{W^{k,\infty}(\Omega;E)} := \max\{\|D^\alpha u\|_{L^\infty(\Omega;E)} \ : \ |\alpha| \le k\}$. In this way $W^{k,p}(\Omega; E)$ is a Banach space. Moreover we set

$$W_0^{k,p}(\Omega; E) := \overline{C_c^\infty(\Omega; E)}$$

where the closure is meant in the topology of $W^{k,p}(\Omega; E)$. If E is reflexive and separable, $p \neq \infty$, and $p' := p/(p-1)$, we set

$$W^{-k,p'}(\Omega; E) := \left(W_0^{k,p}(\Omega; E') \right)'.$$

If $p = 2$ and E is a Hilbert space, then it can be shown that $W^{k,2}(\Omega; E)$ is a Hilbert space when it is endowed with the inner product

$$(u, v)_{W^{k,2}(\Omega;E)} := \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega;E)}.$$

In this case a different notation is used: $H^k(\Omega; E) := W^{k,2}(\Omega; E)$. If $E = \mathbb{R}$ we write $W^{k,p}(\Omega; E) = W^{k,p}(\Omega)$ and $H^k(\Omega; E) = H^k(\Omega)$. $\diamondsuit$

In the sequel of this section we address our attention to Sobolov spaces defined for a special class of domains in $\mathbb{R}^d$.

Definition A.1.5. Let $\Omega \subseteq \mathbb{R}^d$ be an open set. We say that Ω is of *Lipschitz class* if it connected and if there exist two constants $\alpha, \beta > 0$, a finite number $m \in \mathbb{N}_*$ of affine coordinates $(x_{i_1}, \dots, x_{i_d})$ in $\mathbb{R}^d$, $i = 1, \dots, m$, and if there exist m Lipschitz functions $f_i : D_i \longrightarrow \mathbb{R}$, where

$$\hat{x}_i := (x_{i_1}, \dots, x_{i_{d-1}}), \tag{1.1}$$

$$D_i := \{\hat{x}_i \ : \ |x_{i_k}| \le \alpha, \ k = 1, \dots, d-1\}, \tag{1.2}$$

such that the following conditions hold:

$$\partial\Omega = \bigcup_{i=1}^m \{(\hat{x}_i, x_{i_d}) \ : \ \hat{x}_i \in D_i, \ x_{i_d} = f_i(\hat{x}_i)\}, \tag{1.3}$$

$$\{(\hat{x}_i, x_{i_d}) \ : \ \hat{x}_i \in D_i, \ f_i(\hat{x}_i) < x_{i_d} < f_i(\hat{x}_i) + \beta\} \subseteq \Omega, \tag{1.4}$$

$$\{(\hat{x}_i, x_{i_d}) \ : \ \hat{x}_i \in D_i, \ f_i(\hat{x}_i) - \beta < x_{i_d} < f_i(\hat{x}_i)\} \subseteq \mathbb{R}^d \setminus \Omega. \tag{1.5}$$

$\diamondsuit$

Notice that if Ω is of Lipschitz class, then $\mathbf{n}$, the unit outer normal vector to $\partial\Omega$, is defined at $\mathcal{H}^{d-1}$-almost every point of $\partial\Omega$ ($\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure).

Now we state the important Sobolev embedding theorems.

Theorem A.1.1. *Let Ω be an open bounded subset in $\mathbb{R}^d$ of Lipschitz class and let $p, q \in [1, \infty]$, $h, k \in \mathbb{N}$ such that*

$$p \leq q \qquad h \leq k.$$

Then

$$\text{(i)}\ \ h - \frac{d}{q} \leq k - \frac{d}{p} \qquad \Longrightarrow \qquad W^{h,p}(\Omega) \subseteq W^{k,q}(\Omega)\ ,$$

$$\text{(ii)}\ \ k - \alpha \leq h - \frac{d}{p} \qquad \Longrightarrow \qquad W^{h,p}(\Omega) \subseteq C^{k,\alpha}(\Omega),$$

and all the inclusions are continuous. Moreover if the inequalities in (i) *and* (ii) *are replaced by strict inequalities, we have that the inclusions are also compact.*

The following space has been used in Chapter III.

Definition A.1.6. Let Ω be an open subset in $\mathbb{R}^d$. We set

$$L^2_{\mathrm{div}}(\Omega) := \left\{ \mathbf{v} \in L^2(\Omega; \mathbb{R}^d)\ :\ \operatorname{div} \mathbf{v} \in L^2(\Omega) \right\}.$$

The set $L^2_{\mathrm{div}}(\Omega)$ endowed with the inner product

$$(\mathbf{v}_1, \mathbf{v}_2)_{L^2_{\mathrm{div}}(\Omega)} := (\mathbf{v}_1, \mathbf{v}_2)_{L^2(\Omega;\mathbb{R}^d)} + (\operatorname{div} \mathbf{v}_1, \operatorname{div} \mathbf{v}_2)_{L^2(\Omega)}, \qquad \mathbf{v}_1, \mathbf{v}_2 \in L^2_{\mathrm{div}}(\Omega)$$

is a Hilbert space. ♢

Now we need to define Sobolev spaces where the esponent k is allowed to be noninteger. We confine to the case of real-valued functions.

Definition A.1.7. Let Ω be an open subset of $\mathbb{R}^d$ and let $s \in]0, \infty[\setminus \mathbb{N}$. If we set $\sigma := s - [s]$ (where $[s]$ is integer part of s), then for any $u \in L^1(\Omega)$ and for any multiindex α we define the function $r^\alpha_\sigma : \Omega \times \Omega \longrightarrow \mathbb{R}$ by

$$(r^\alpha_\sigma(u))(x, y) := \frac{D^\alpha u(x) - D^\alpha u(y)}{|x - y|^{\sigma + n/2}}.$$

The Sobolev space $H^s(\Omega)$ is defined by

$$H^s(\Omega) := \left\{ u \in L^1(\Omega) \ : \ u \in H^{[s]}(\Omega),\ r^\alpha_\sigma(u) \in L^2(\Omega \times \Omega) \right\},$$

and we endow it with the inner product

$$(u,v)_{H^s(\Omega)} := (u,v)_{H^s(\Omega)} + \sum_{|\alpha|=[s]} (r^\alpha_\sigma(u), r^\alpha_\sigma(v))_{L^2(\Omega\times\Omega)}.$$

If $s \in]-\infty, 0[$ then we define $H^s(\Gamma) := (H^{-s}(\Gamma))'$. $\diamondsuit$

Now we can define the Sobolev spaces on boundaries.

Definition A.1.8. Let $\Omega \subseteq \mathbb{R}^d$ be bounded and of Lipschitz class. Let denote by Γ its boundary, and let $s > 0$. Then using the same notation of Definition A.1.5 we define

$$H^s(\Gamma) := \left\{ u \in L^2(\Gamma) \ : \ \hat{x}_i \mapsto u(\hat{x}_i, f_i(\hat{x}_i)) \in H^s(D_i),\ i = 1, \dots, m \right\}.$$

We endow this space with the norm

$$\|u\|_{H^s(\Gamma)} := \left(\sum_{i=1}^d \|u(\cdot, f_i(\cdot))\|^2_{H^s(D_i)} \right)^{1/2}.$$

Finally we set $H^{-s}(\Gamma) := (H^s(\Gamma))'$. $\diamondsuit$

The spaces $H^s(\Gamma)$ allow to define traces of functions on the boundary.

Theorem A.1.2. *Let Ω be an open bounded and connected subset of $\mathbb{R}^d$ having a Lipschitz boundary Γ. Then there exists a unique $\gamma_\Gamma \in \mathscr{L}(H^1(\Omega), H^{1/2}(\Gamma))$ which is surjective and such that $\gamma_\Gamma(\varphi) = \varphi|_\Gamma$ for all $\varphi \in C^\infty(\overline{\Omega})$.*

Theorem A.1.3. *Let Ω be an open bounded and connected subset of $\mathbb{R}^d$ having a Lipschitz boundary Γ. Let $\mathbf{n}$ be the unit outer normal vector to Γ. Then there exists a unique $\gamma_\mathbf{n} \in \mathscr{L}(L^2_{\mathrm{div}}(\Omega), H^{-1/2}(\Gamma))$ which is surjective and such that $\gamma_\mathbf{n}(\boldsymbol{\varphi}) = \boldsymbol{\varphi}|_\Gamma \cdot \mathbf{n}$ for all $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$. Moreover if $u \in H^1(\Omega), \mathbf{v} \in L^2_{\mathrm{div}}(\Omega)$, the following Green formula holds:*

$$\int_\Omega \nabla u \cdot \mathbf{v} = -\int_\Omega u \operatorname{div} \mathbf{v} + {}_{H^{-1/2}(\Gamma)}\langle \gamma_\mathbf{n}(\mathbf{v}), u \rangle_{H^{1/2}(\Gamma)} \tag{1.6}$$

Definition A.1.9. Let Ω be an open bounded and connected subset of $\mathbb{R}^d$ that we assume to be of Lipschitz class. Let Γ be the boundary of Ω. Let Γ_D an open connected subset of Γ and let $\Gamma_N := \Gamma \setminus \overline{\Gamma}_D$ having $(d-1)$-dimensional Hausdorff measure strictly greater than zero. The definition of the space $H^s(\Gamma_D)$ is similar to that of $H^s(\Gamma)$. Then we denote by $H^s_0(\Gamma_D)$ the closure in $H^s(\Gamma)$ of the set

$$\left\{ v|_\Gamma \;:\; v \in C^\infty(\overline{\Omega}),\ v \text{ vanishes in some neighborhood of } \Gamma \setminus \Gamma_D \right\}$$

Finally we set $H^{-s}(\Gamma_D) := (H^s_0(\Gamma_D))'$. $\diamondsuit$

Definition A.1.10. Let Ω be an open bounded and connected subset of $\mathbb{R}^d$ of Lipschitz class and boundary Γ. Let Γ_D an open connected subset of Γ and let $\Gamma_N := \Gamma \setminus \overline{\Gamma}_D$ having $(d-1)$-dimensional Hausdorff measure strictly greater than zero. Let us denote by $\widetilde{u}$ the extension to 0 of u on Γ. We set

$$H^s_{00}(\Gamma_D) := \{ u \in H^s(\Gamma_D) \;:\; \widetilde{u} \in H^s(\Gamma) \}$$

and we endow this space with the norm $\|u\|_{H^s_{00}(\Gamma_D)} := \|\widetilde{u}\|_{H^s(\Gamma)}$. $\diamondsuit$

Assume the hypoteses of Definition A.1.10. Then if $u \in H^{1/2}(\Gamma)$, its restriction $u|_{\Gamma_D}$ in general does not belong to $H^{1/2}(\Gamma_D)$. However it makes sense the following definition.

Definition A.1.11. Assume the hypoteses of Definition A.1.10. Given a function $u \in H^{1/2}(\Gamma)$ we define the functional $\gamma_{\Gamma_D}(u) \in (H^{1/2}_{00}(\Gamma_D))'$ by

$${}_{(H^{1/2}_{00}(\Gamma_D))'}\langle \gamma_{\Gamma_D}(u), v \rangle_{H^{1/2}_{00}(\Gamma_D)} := (u, \widetilde{v})_{H^{1/2}(\Gamma)} \qquad \forall v \in H^{1/2}_{00}(\Gamma_D).$$

$\diamondsuit$

Definition A.1.12. Let Ω be an open bounded and connected subset of $\mathbb{R}^d$ having Lipschitz boundary Γ. Let Γ_D an open connected subset of Γ. We define

$$H^1_{\Gamma_D}(\Omega) := \left\{ u \in H^1(\Omega) \;:\; \gamma_{\Gamma_D}(u) = 0 \right\}.$$

$\diamondsuit$

Proposition A.1.2. *Under the assumptions of Definition A.1.10 we have*

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) \ : \ \gamma_\Gamma(u) = 0\right\}, \tag{1.7}$$

$$H_{00}^{1/2}(\Gamma_N) = \left\{\gamma_0(u) \ : \ u \in H^1_{\Gamma_D}(\Omega)\right\}. \tag{1.8}$$

Remark A.1.1. Notice that under the assumptions of Definition A.1.10, if $u \in L^2_{\mathrm{div}}(\Omega)$ then the trace of $\gamma_{\mathbf{n}}(u)$ on Γ_D in general does not belong to $H^{-1/2}(\Gamma_D)$, but it is contained in a larger space, namely $(H_{00}^{1/2}(\Gamma_D))'$. See [24] for more on these spaces.

A.2 Abstract functions

Let E be a Banach space and $T > 0$. In this section we will consider maps defined on the interval $[0, T]$ with values in E, the so called abstract functions. We will use the dot notation to denote the (Fréchet) derivative, i.e. the limit of the incremental ratio, when this exists; to be more precise if $u : [0, T] \longrightarrow E$ is differentiable in a point t, the symbol $\dot{u}(t)$ will represent the derivative of u in this point.

Definition A.2.1. We say that a map $u : [0, T] \longrightarrow E$ is *absolutely continuous (on* $[0, T]$*)* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}_*$ and for every family $(]a_i, b_i[)_{i=1}^n$ of mutually disjoint intervals in $[0, T]$, the following condition holds:

$$\sum_{i=1}^n (b_i - a_i) < \delta \quad \Longrightarrow \quad \sum_{i=1}^n \|u(b_i) - u(a_i)\|_E < \varepsilon.$$

The set of such absolutely continuous maps is denoted by $AC(0, T; E)$. ♢

An important class of absolutely continuous maps is given by the integral functions, as shown by the following

Proposition A.2.1. *Let* $w \in L^1(0, T; E)$*,* $t_0 \in [0, T]$*, and define the mapping* $I_{t_0}w : [0, T] \longrightarrow E$ *by*

$$(I_{t_0}w)(t) := \int_{t_0}^t u(s)ds, \qquad t \in [0, T].$$

Then $I_{t_0}w \in AC(0, T; E)$*,* $I_{t_0}w$ *is differentiable in almost every* $t \in [0, T]$*, and its (Fréchet) derivative is almost everywhere equal to* u*.*

In general E-valued absolutely continuous maps of one variable cannot be represented as a "primitive function", at variance with the finite dimensional case, i.e. if $u \in AC(0,T;E)$, then there does not necessarily exist a function $w \in L^1(0,T;E)$ such that $u(t) = u(0) + \int_0^t w(s)ds$ for all $t \in [0,T]$. A sufficient condition for the existence of such a representation is given by the following

Theorem A.2.1. *Assume that E is reflexive and that $u \in AC(0,T;E)$. Then there exists $\dot{u}(t)$ for almost every $t \in [0,T]$, $\dot{u} \in L^1(0,T;E)$, and*

$$u(t) = u(0) + \int_0^t \dot{u}(s)ds \qquad \forall t \in [0,T].$$

If E is not a reflexive space, in general Theorem A.2.1 does not hold, a counterexample being given by $u : [0,T] \longrightarrow L^1(0,1)$, $u(t) := \chi_{[0,t]}$, which is absolutely continuous, but nondifferentiable in any point (see [4, p. 15]). This, thanks to Proposition A.2.1, clearly entails that in this case $AC(0,T;E)$ properly contains the set of maps having an integral representation. Next theorem shows that such set is essentially given by the Sobolev space $W^{1,1}(0,T;E)$, i.e. the space of all maps $u \in L^1(0,T;E)$ having their distributional derivative $u' := D^1 u$ belonging to $L^1(0,T;E)$.

Theorem A.2.2. *Let $p \geq 1$. Then the following propositions are equivalent.*

(i) $u \in W^{1,p}(0,T;E)$

(ii) $\exists u_0 \in E \;:\; u(t) = u_0 + \int_0^t u'(s)ds$ *for a.a.* $t \in]0,T[$

(iii) *For each $v^* \in E'$ let $\phi :]0,T[\longrightarrow \mathbb{R}$ be the function defined by $\phi(t) := {}_{E'}\langle v^*, u(t)\rangle_E$. Then $\phi \in W^{1,p}(0,T)$ and its distributional derivative is given by $\phi'(t) = {}_{E'}\langle v^*, u'(t)\rangle_E$*

Proposition A.2.2. *Let (V,H,V') a Hilbert triplet and let $u \in L^2(0,T;V)$ (respectively $u \in L^2(0,T;V) \cap L^\infty(0.T;H)$) such that $u' \in L^2(0,T;V')$ (respectively $u' \in L^2(0,T;V') + L^1(0.T;H)$). Then u is almost everywhere equal to a continuous H-valued function on $[0,T]$. In other terms the following inclusions holds*

$$L^2(0,T;V) \cap H^1(0,T;V') \subseteq C([0,T];H), \tag{2.1}$$

$$L^2(0,T;V) \cap L^\infty(0,T;H) \cap \left[H^1(0,T;V') + W^{1,1}(0,T;H)\right] \subseteq C([0,T];H). \tag{2.2}$$

Moreover the previous inclusions are continuous.

Notice that from (2.1)–(2.2) it follows that $L^2(0,T;V) \cap H^1(0,T;V')$ is contained in $L^2(0,T;V) \cap L^\infty(0,T;H) \cap [H^1(0,T;V') + W^{1,1}(0,T;H)]$.

Proposition A.2.3. *Let (V,H,V') a Hilbert triplet and let $u,v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that $u,v \in L^2(0,T;V) + L^1(0,T;H)$. Then the function $\phi : t \mapsto (u(t),v(t))_H$ is absolutely continuous on $[0,T]$, and*

$$\phi'(t) = {}_{V'}\langle u'(t), v(t)\rangle_V + {}_{V'}\langle v'(t), u(t)\rangle_V \qquad \textit{for a.a. } t \in]0,T[,$$

$$\int_s^t {}_{V'}\langle u'(\tau), v(\tau)\rangle_V d\tau = (u(t),v(t))_H - (u(s),v(s))_H - \int_s^t {}_{V'}\langle v'(\tau), u(\tau)\rangle_V d\tau \qquad \forall s,t \in [0,T].$$

Corollary A.2.1. *Let (V,H,V') a Hilbert triplet and let $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that $u \in L^2(0,T;V) + L^1(0,T;H)$. Then the function $\psi : t \mapsto \|u(t)\|_H^2$ is absolutely continuous on $[0,T]$, and*

$$\psi'(t) = 2v_{V'}\langle u'(t), u(t)\rangle_V \qquad \textit{for a.a. } t \in]0,T[,$$

$$2\int_s^t v_{V'}\langle u'(t), u(t)\rangle_V = \|u(t)\|_H^2 - \|u(s)\|_H^2 \qquad \forall s,t \in [0,T].$$

Proposition A.2.4. *Consider E, F Banach spaces with $E \subset F$ with continuous embeddings. Let $p,q \in [1,\infty[$ and let (u_n) be a sequence in $L^p(0,T;E)$ such that $u_n' \in L^q(0,T;F)$ for all n. Assume that $u_n \rightharpoonup u$ in $L^p(0,T;E)$ and $u_n' \rightharpoonup v$ in $L^q(0,T;F)$. Then $u' = v$.*

It is also useful to consider the space $C_w(0,T;E)$ of all continuous maps from $[0,T]$ in E, where E is a Banach space endowed with the weak topology. We have the following Proposition.

Proposition A.2.5. *Consider E, F Banach spaces with $E \subseteq F$ with continuous embeddings. Then*

$$L^\infty(0,T;E) \cap C_w(0,T;F) = C_w(0,T;E).$$

The proof can be found in [24].

We finish with a property of the operator A used throughuot the dissertation. We recall its definition. Under the assumption of Definition A.1.10 in the previous section, if $V := H^1_{\Gamma_D}(\Omega)$, the operator $A \in \mathscr{L}(V, V')$ is defined by

$$_{V'}\langle Av_1, v_2\rangle_V := \int_\Omega \nabla v_1 \cdot \nabla v_2, \qquad v_1, v_2 \in V.$$

We have the following proposition that can be proved by means of a regularization procedure

Proposition A.2.6. *Under the assumption of Definition A.1.10, let $T > 0$, $H := L^2(\Omega)$, and $u, v \in L^2(0,T;V) \cap H^1(0,T;H)$. Let us suppose also that $Au, Av \in L^2(0,T;H)$. Then the following formula is valid for all $s, t \in [0,T]$, with $s \leq t$.*

$$\begin{aligned} &\int_s^t (Au(\tau), v'(\tau))_H d\tau \\ &= {}_{V'}\langle Au(t), v(t)\rangle_V - {}_{V'}\langle Au(s), v(s)\rangle_V - \int_s^t (Av(\tau), u'(\tau))_H d\tau. \end{aligned}$$

This property holds for an abstract Hilbert triplet and for a general operator A having a suitable coercivity property.

A.3 Compactness

In this section we state two compactness theorems that are widely used in partial differential equations. The first theorem is a generalized version of the classical theorem due to Ascoli. Its proof can be found, e.g., in [22].

Theorem A.3.1 (Ascoli)**.** *Let (K, d) be a compact metric space and let E be Banach space. Let $\mathscr{F}$ a subset of $C(K;E)$, the space of continuous maps from K into E, endowed with sup norm. Then $\mathscr{F}$ is precompact in $C(K;E)$ if and only if the two following conditions hold.*

(i) *$\mathscr{F}$ is equicontinuous, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathscr{F}$ we have $\|f(x) - f(y)\|_E < \varepsilon$ whenever $d(x,y) < \delta$.*

(ii) *For each $x \in K$ the set $\{f(x) \ : \ f \in \mathscr{F}\}$ is precompact in E.*

Now we present another criterion whose proof is contained in [23].

Theorem A.3.2 (Aubin Lemma)**.** *Let E, E_1, and E_2 be Banach spaces. Assume that E_1 and E_2 are reflexive and*

$$E_1 \subseteq E \subseteq E_2,$$

where the former embedding is compact and the latter is continuous. If $p, q \in]1, \infty[$ and $T > 0$, then the space

$$W := \{u \in L^p(0,T;E_1) \ : \ u' \in L^q(0,T;E_2)\},$$

endowed with the norm

$$\|u\|_W := \|u\|_{L^p(0,T;E_1)} + \|u'\|_{L^q(0,T;E_2)}, \qquad u \in W,$$

is compactly embedded in $L^p(0,T;E)$.

Other compactness theorems having useful applications to partial differential equations are contained in the paper [34].

A.4 Fixed point theorems

Throughout the dissertation we make an extensive use of some fixed point theorems. We state here these theorems. The first two results are classical.

Theorem A.4.1 (Banach)**.** *Let (E,d) a complete metric space, and let $\Sigma : E \longrightarrow E$ be a mapping. Assume that there exists a constant $L \in]0,1[$ such that*

$$d(\Sigma(x), \Sigma(y)) \leq Ld(x,y) \qquad \forall x, y \in E$$

(i.e. Σ is a (strict) contraction). Then Σ has a unique fixed point z, that is a point such that $\Sigma(z) = z$.

The following corollary is a straightforward consequence of the Banach fixed point Theorem.

Corollary A.4.1 (Banach)**.** *Let (E,d) a complete metric space, and let $\Sigma : E \longrightarrow E$ be a mapping. Assume that there exist a positive integer $m \in \mathbb{N}_*$ and a constant $L \in]0,1[$ such that*

$$d(\Sigma^m(x), \Sigma^m(y)) \leq Ld(x,y) \qquad \forall x, y \in E.$$

Then Σ has a unique fixed point z.

Since we deal with multivalued maps, it is convenient to have at disposal a thereom about "fixed point" of such maps. The meaning of fixed point for a multifunction is clarified by the following definition.

Definition A.4.1. Let E be a set and let $\Sigma : E \longrightarrow \mathscr{P}(E)$ a multivalued map. An element $z \in E$ is called a *fixed point* of Σ if $z \in \Sigma(z)$. $\diamondsuit$

The theorem that we are going to present is due to Gliksberg (cf. [19]).

Theorem A.4.2. *Let E be a Hausdorff locally convex topological vector space and K a nonempty compact and convex subset of E. Suppose that $\Sigma : K \longrightarrow \mathscr{P}(K)$ satisfies the conditions*

$$\Sigma(x) \text{ is a nonempty, closed, and convex set} \quad \forall x \in K, \tag{4.1}$$

$$G_{\mathcal{R}}(\Sigma) := \{(x,y) \in K \times K \ : \ y \in \Sigma(x)\} \text{ is closed in } K \times K. \tag{4.2}$$

Then Σ has at least a fixed point.

A.5 Gronwall lemma

Theorem A.5.1. *Let E be a Banach space and let $g : E \times I \longrightarrow E$ such that $g(\cdot, t)$ is Lipschitz continuous for all $t \in I$ uniformly with respect to the first variable, and $g(x, \cdot) \in L^1(I)$ for all $x \in E$.*

An essential tools in ordinary differential equations is the Gronwall Lemma.

Proposition A.5.1 (Gronwall)**.** *Let $m \in L^1(0,T)$ such that $m \geq 0$ a.e. in $]0,T[$ and let $a \geq 0$. If $\phi \in C([0,T])$ satisfies the inequality*

$$\phi(t) \leq a + \int_0^t m(s)\varphi(s)ds \qquad \forall t \in [0,T],$$

then

$$\phi(t) \leq a \exp\left(\int_0^t m(s)ds\right) \qquad \forall t \in [0,T].$$

We will often use an extended form of the Gronwall Lemma, whose proof can be achieved combining the two lemmas in [5, Lemma A.4, Lemma A.5, pp. 156-157] (see also [3], where a more general version is proved).

Proposition A.5.2. *Let $m \in L^1(0,T)$ such that $m \geq 0$ a.e. in $]0,T[$ and let $a, b \geq 0$. If $\phi \in C([0,T])$ is such that $\phi \geq 0$ on $[0,T]$ and satisfies the inequality*

$$\frac{1}{2}\left(\phi(t)\right)^2 \leq \frac{1}{2}a^2 + b^2 \int_0^t \left(\varphi(s)\right)^2 ds + \int_0^t m(s)\varphi(s)ds \qquad \forall t \in [0,T],$$

then

$$\left(\phi(t)\right)^2 \leq \left(2a^2 + \frac{1}{2}\|m\|^2_{L^1(0,T)}\right) e^{2b^2 t} \qquad \forall t \in [0,T].$$

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